THE TWENTY-NINTH SUMMER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY

The twenty-ninth summer meeting of the Society, consisting of two sessions, was held at the University of Rochester, Rochester, New York, Thursday and Friday, September 7–8, 1922, in conjunction with the meeting of the Mathematical Association of America. Many members of the Society attended the special session of the Association on Thursday morning, held at the Research Laboratory of the Eastman Kodak Company, and members of both organizations were invited to visit the buildings of the Bausch and Lomb Optical Company on Friday afternoon. A much appreciated feature of the meeting was the chamber concert given in Kilbourn Hall of the new Eastman School of Music which is an integral part of the University of Rochester. The joint dinner, preceded by a very enjoyable automobile ride arranged by the alumni of the University, was held on Thursday evening at the Oak Hill Country Club, on the future site of the University. At this dinner Professor R. C. Archibald, President of the Association and Librarian of the Society, gave an interesting talk on his impressions of European conditions.

The attendance at the scientific sessions included the following seventy members of the Society:

Archibald, W. J. Berry, William Betz, Borger, W. G. Bullard, R. W. Burgess, Cairns, W. B. Carver, Clarke, H. E. Cobb, Coble, Comstock, Copeland, Crathorne, Decker, Durfee, Eisenhart, Ettlinger, Finkel, Fischer, W. B. Ford, Gale, Gilman, Gummer, Harrell, E. R. Hedrick, Hildebrandt, Hurwitz, Ingels, Karpinski, Kellogg, Kuhn, W. D. Lambert, Lefschetz, Lennes, Long, Matheson, G. A. Miller, Norman Miller, Mirick, C. N. Moore, F. H. Murray, F. W. Owens, H. B. Owens, Pitcher, Plant, Rasor, F. W. Reed, Reid, Harris Rice, R. G. D. Richardson, Ritt, E. D. Roe, J. R. Roe, Seely, Sharpe, Sherk, W. G. Simon, Slaught, C. E. Smith, Virgil Snyder, Swartzel, Watkeys, J. H. Weaver, Wedderburn, D. E. Whitford, F. B. Williams, W. L. G. Williams, Yeaton, J. W. Young.

The Secretary announced the election of the following persons to membership in the Society:

Dr. Beulah May Armstrong, University of Illinois;
Mr. Santiaga Artiaga, City Engineer, Manila;
Professor Francis Easton Carr, Oberlin College;
Dr. Vishnu D. Gokhale, University of Chicago;
Dr. Charles Jordan, Budapest;
Professor Henri Lebesgue, Collège de France;
Mr. Earle Brenneman Miller, University of Wisconsin;
Mr. Edward Everett Rhodes, Mutual Benefit Life Insurance Company;
Mr. Harvey Alexander Simmons, University of Michigan;
Dr. Ernest Bloomfield Zeisler, University of Chicago.

At the meeting of the Council, twenty-one applications for membership in the Society were received.

The Council announced the appointment of the following committees: Professors Haskins, Fiske and H. S. White on award of the Bôcher Memorial Prize; Professors Curtiss, Dunham Jackson and Mitchell on nomination of officers; Professors Coolidge, Fischer, Kellogg, Roever, Tyler, and the Secretary of the Society, on arrangements for the annual meeting; Professors Fite and Dresden and Mr. Joffe on investment of the Eliakim Hastings Moore Fund.

It was announced that the extra volume of the TRANSACTIONS is being printed by the Eschenbach Company, of Easton, Pa.

Greetings were sent to the newly-formed Mathematical Society of Belgium.

At the meeting of the Society, the reciprocity agreement with the London Mathematical Society was ratified, on recommendation of the Council. The medal presented to the Society by the l'Académie royale des Sciences et Beaux-Arts de Belgique, on the occasion of its one hundred fiftieth anniversary, was exhibited. A resolution was passed thanking the University of Rochester, and in particular the department of mathematics, for its generous hospitality.

At the sessions of the Society Professor A. B. Coble presided, relieved by Professors L. P. Eisenhart, G. A. Miller and Virgil Snyder. The session of Friday morning was especially marked by a paper read, at the request of the programme committee, by Professor C. A. Fischer, on *Functions of lines*.

Titles and abstracts of the papers read at this meeting follow below. The papers of Professors Cole, R. L. Moore

and Kline, Dr. Schmidt, Professor Myller, Dr. Zeldin, Professor Graustein, Miss Carlson, Professor Evans, Dr. Franklin, Dr. Murray, Professor Dodd, and the third paper by Professor Schwatt, were read by title. Dr. Camp's paper was read by Professor C. N. Moore, Mr. Cowling's by Professor Ettlinger, Professor Schwatt's first paper by Professor Snyder, and his second by Professor Ettlinger. Mr. Cowling was introduced by Professor Ettlinger.

1. Professor L. P. Eisenhart: *Condition that a tensor be the curl of a vector.*

This paper will appear in the December number of this BULLETIN.

2. Professor S. Lefschetz: *A new class of topological invariants for two-sided manifolds.*

In a two-sided M_n consider a fundamental system Γ_k^1, Γ_k^2, . . . for the k-cycles, $k \leqq \frac{1}{2}n$, and a similar one Γ_{n-k}^1, Γ_{n-k}^2, . . . for the $(n-k)$-cycles; then denote by $(\Gamma_k^i \Gamma_{n-k}^j)$ the number of intersections of the two cycles when a definite sign is attached to each point of intersection after the manner of Poincaré. The elementary divisors of the matrix

$$\|(\Gamma_k^i \Gamma_{n-k}^j)\|$$

are the invariants in question. It is readily seen that for an M_2 they are all equal to unity. Remarkably enough this is also true for an algebraic surface (though not for the most general M_4), also for a d-dimensional algebraic variety and $k = 1, 2$. As the topological invariants already known have rather arbitrary values for algebraic surfaces, it would seem that their relations with those just defined are unlikely to be simple.

3. Professors F. R. Sharpe and Virgil Snyder: *The* (1, 2) *quaternary correspondences associated with certain space involutions.*

In this paper Professors Sharpe and Snyder use the complex of lines determined by the pairs of conjugate points of a given space involution to find the (1, 2) quaternary correspondence associated with the involution. The method can be applied

to known involutions that are not readily discussed by the inverse process. The non-monoidal cubic involution is proved to be rational. The special case of the cubic inversion is of particular interest because of the complicated configuration of the fundamental elements.

4. Professor C. N. Moore: *On the summability of the double Fourier's series.*

The principal result of this paper is the theorem that the double Fourier's series corresponding to a function of two variables $f(x, y)$ that is integrable (Lebesgue) in a certain region of the (x, y)-plane is summable $(C1)$ to the value of the function throughout the region, except for a set of points of measure zero. This is an extension to double Fourier's series of a theorem with regard to ordinary Fourier's series due to Lebesgue. A further result of the paper is an extension to double Fourier's series of Fejér's theorem under wider conditions than have hitherto been found.

5. Professor N. J. Lennes: *The theory of sets and the foundation of arithmetic.*

The commutative and associative laws of addition and multiplication as well as the distributive law of multiplication are easily shown to hold under any set of axioms sufficient for the ordinary theory of sets. In the last analysis, integers may be regarded as counters for "things," and hence it is proposed to make the properties of integers under the operations of addition and multiplication depend logically upon the corresponding properties of sets.

This paper gives an outline of the process by which the usual axioms of arithmetic may be derived as theorems from the corresponding propositions in the theory of sets.

6. Professor F. N. Cole: *Kirkman parades.*

This paper will appear in the December number of this BULLETIN.

7. Professors R. L. Moore and J. R. Kline: *On the definition of a simple closed surface.*

The authors suggest the following definition for a simple closed surface: A simple closed surface is a closed, bounded and connected point set M such that (a) every simple closed

curve J which belongs to M is the common boundary, with respect to M, of two mutually separated connected subsets of M whose sum is $M - J$. (*b*) If P is a point of M and ϵ is a positive number, there exists in M a simple closed curve J such that, of the two mutually separated connected sets into which M is divided by the omission of J, one contains P and is of diameter less than ϵ. If in a euclidean space of three dimensions M is a point set satisfying this definition, then the authors prove that there is a one-to-one continuous correspondence between the point set M and the surface of a sphere.

8. Dr. Karl Schmidt: *The theory of functions of one Boolean variable.*

In this paper, the expression $f(x) = ax + b\bar{x}$ of the Boolean algebra, or algebra of logic, is studied by new methods, which lead to new results and throw new light on old results. The principal references are to the works of E. Schröder and Eugen Müller. The paper will be published in the TRANSACTIONS OF THIS SOCIETY.

9. Professor A. Myller: *Representation of rectilinear motion by the geodesics of a surface.*

Professor Myller shows that the world-lines, in the Minkowski sense, of any rectilinear motion are in one-to-one correspondence with the geodesics of a surface of revolution, and conversely.

10. Dr. S. D. Zeldin: *Note on steady fluid motion.*

This paper appeared in full in the July number of this BULLETIN.

11. Professor W. C. Graustein: *Real representations of analytic complex curves.*

In a previous paper, presented to the Society in December, 1919, all the representations of a complex point by pairs of ordered real points which satisfy certain general conditions were determined. In the present paper application is made to analytic complex curves, in the plane and in three-dimensional space, of certain of these representations, namely, those invariant under direct transformations of similarity.

12. Miss Elizabeth Carlson: *Extension of Bernstein's theorem to Sturm-Liouville sums.*

The following theorem is proved in this paper: The maxi-

mum of the absolute value of the derivative of a Sturm-Liouville sum of order n does not exceed nhM, where M is the maximum of the absolute value of the sum itself, and h is a constant independent of n and of the coefficients in the sum. The corresponding theorem for finite trigonometric sums is due to S. Bernstein.

13. Professor G. C. Evans: *A Bohr-Langmuir contact transformation.*

The author considers the relation between the Bohr and Langmuir atoms, the latter having a quantized force, with regard to the question as to whether the "events" of one system may be transformed into the events of the other. It is shown that one cannot be the transform of the other by any contact transformation strictly so-called, which does not involve the time explicitly, but that the Langmuir atom can be obtained from the Bohr atom by a transformation which does not involve the time and is a contact transformation "im kleinen," although not "im grossen." The vibrating Langmuir atom is seen to be equivalent to the Bohr atom with elliptical electron orbit; the completely static atom is equivalent to the Bohr atom with circular orbit.

14. Professor G. C. Evans: *An elementary theory of competition.*

The author discusses postulates for monopoly, cooperation and two different sorts of competition, and analyzes them with reference to a simple economic system. The order of prices is obtained, and the effects of price fixing, of excess profits taxation and of certain typical changes in the cost function.

15. Professor G. A. Miller: *Groups in which the number of operators in a set of conjugates is equal to the order of the commutator subgroup.*

The author first proves that every abelian group can be used as a commutator subgroup of some group which involves a set of conjugate operators whose number is equal to the order of the commutator subgroup. He then determines the properties of the system of groups such that every group of the system contains a set of conjugate operators whose number is equal to the order of the commutator subgroup diminished

by one. In particular, he proves that every group of such a system must also contain a set of conjugate operators whose number is equal to the order of the commutator subgroup, and that this commutator subgroup is always an abelian prime power group of type $(1, 1, 1, \cdots)$.

16. Dr. C. C. Camp: *Expansions in terms of solutions of partial differential equations. First paper: Multiple Fourier expansions.*

By assuming u to be a product of functions of the separate variables the author finds a solution of $\Sigma_{i=1}^{p}\partial u/\partial x_i + \lambda u = 0$ for the region $-\pi \leqq x_i \leqq \pi$ and the boundary conditions which make u identical for both ends of the several x_i-intervals. This depends on the solution of p ordinary differential equations with boundary conditions, each containing a different parameter. The expansion of a function f in p variables, which is made up of a finite number of pieces, each real, continuous, and possessing continuous partial derivatives, involves solutions of the adjoint systems and is shown by an extension of Professor Birkhoff's method to converge uniformly to f inside the subregion of any such piece. At a point of discontinuity the series, which is always expressible as a multiple Fourier series, converges to the mean value. By a suitable interpretation this is also true on the boundaries. This is believed to be the first adequate proof of convergence for a multiple Fourier series in more than two variables.

17. Professor E. H. Clarke: *On the minimum of the sum of a definite integral and a function of a point.*

Hadamard, in his *Leçons sur le Calcul des Variations*, has given, in § 159, pp. 176–177, a short notice of this problem. It may be stated briefly as follows: Given two points P_1 and P_2 in the xy plane and two functions $\varphi(x, y)$ and $f(x, y, y')$ of class C_{111}, to find a curve C, joining P_1 to P_2, and a point $\pi(x, y)$, lying on the curve C, such that the sum of the numerical value of the integral of $f(x, y, y')$ taken along C and the numerical value of the function $\varphi(x, y)$ for the point π shall be as small as possible. It is proved in this paper that in addition to the usual necessary conditions of the calculus of variations for the arcs $\rho_1\pi$ and $\pi\rho_2$ there are certain other necessary conditions which must hold at the point π and for the family of broken extremals with corners on a curve K through π. These conditions, when slightly strengthened, are

shown to be sufficient to insure the existence of a minimum for the above sum.

18. Professor H. J. Ettlinger: *A simple proof of a fundamental lemma concerning the limit of a sum.*

The author gives a proof of the following lemma upon which may be based the theory of the Riemann integral.

Let the interval $I : a \leqq x \leqq b$ be divided into n equal sub-intervals, $I(i, n)$, each of length $\Delta_n x = (b - a/n)$. To each $I(i, n)$ there corresponds a number $h(i, n)$. Let $|h(i, n)| \leqq M$ for all values of $i \leqq n$ and n, where M is a constant. If x is any fixed value in I, then for each value of n, x is contained in at least one of the sub-intervals. Let it be designated by $I(x, n)$. For each fixed x and each subdivision $I(x, n)$ let the corresponding number be $h(x, n)$, and let $\lim_{n\to\infty} h(x, n) = 0$. Then $\lim_{n\to\infty} \sum_1^n h(i, n)\Delta_n x = 0$.

The proof makes use of simple limit theorems such as are to be found in every calculus textbook.

19. Mr. A. H. Cowling: *Application of Duhamel's theorem to the convergence proof for approximate solutions of differential equations.*

Runge, Heun and Kutta have given certain formulas for the approximate solution of the differential equation $y' = f(x, y)$. Assuming the conditions for which the Cauchy polygon converges, Mr. Cowling shows that these approximations approach the true solution as a limit. In this proof, use is made of Duhamel's theorem in the form given by Professor Ettlinger (see forthcoming paper in the AMERICAN MATHEMATICAL MONTHLY).

20. Dr. Philip Franklin: *Two theorems on multiple integrals.*

This paper will appear in the December number of this BULLETIN.

21. Professor E. L. Dodd: *An extension of the theorem of Bayes, by the use of a certain limit.*

If an observed event E must have been preceded by one of the mutually exclusive conditions C_i, $i = 1, 2, \cdots$; and $f(i)$ is the *a priori* probability that C_i would obtain; and $p(i)$ is

the "productive" probability that C_i would yield E; then the *a posteriori* probability $P(r)$ that C_r preceded E is $f(r)p(r) \div \Sigma f(i)p(i)$. If, however, $f(x)dx$ is the probability that an error will lie between x and $x + dx$, and all real values are postulated as equally likely *a priori*, then $f(x) \equiv 0$,—if, indeed, it can be defined at all. The following extension is suggested to meet this difficulty:

$$P(r) = \lim_{t \to \infty} \frac{f(r, t)p(r)}{\sum_{i=-t}^{i=t} f(i, t)p(i)}.$$

Integration may replace summation. The author gives an example justifying this extension.

22. Dr. W. L. G. Williams: *Fundamental systems of protomorphic formal modular seminvariants of binary forms.*

The author defines fundamental systems of formal modular seminvariants, modulo p, of one binary form and of several binary forms and finds fundamental systems in both cases. These are remarkable in that each of them contains only one formal modular seminvariant not congruent, modulo p, to an algebraic seminvariant. Sufficient conditions for the existence of certain algebraic invariants are derived from the theory of formal modular concomitants and a connection with Hermite's law of reciprocity is noted.

23. Professor O. D. Kellogg: *An example in potential theory.*

The question of the extreme of generality of which the Dirichlet principle is susceptible is closely connected with the question of finite discontinuities of harmonic functions, where by discontinuity is meant a discontinuity either of the function or of a first derivative. It is well known that a harmonic function of x and y cannot have a finite non-removable discontinuity at an isolated point. Professor Kellogg gives a typical example of a harmonic function having finite non-removable discontinuities at every point of a nowhere dense perfect point set of measure zero. Green's function for the infinite plane, bounded by this set, is also established.

24. Mr. F. H. Murray: *On certain systems of differential equations containing a parameter.*

Systems of differential equations, of the form

$$dx_i/dt = X_i(x_1, \cdots, x_n, v, t), \qquad (i = 1, 2, \cdots, n)$$

where X_i is analytic in v in the neighborhood of v_0 and satisfies the usual conditions with respect to $x_1, \cdots, x_n, t$ in a certain domain $|t - t_0| < a$, $|x_i - x_i^0| < b$, have been discussed by Poincaré, Picard, and others. In this paper the writer proposes to give a similar discussion of the system

$$dx_i/dt = X_i(x_1, \cdots, x_n, w)$$

where X_i satisfies the usual conditions with respect to $x_1, \cdots, x_n$, and is periodic of period 2π with respect to the variable $w = vt$, for large values of v. An application is made to a problem related to the restricted problem of three bodies.

25. Mr. F. H. Murray: *Periodic solutions in the problem of three bodies.*

In *Les Méthodes Nouvelles de la Mécanique Céleste*, Poincaré gave a discussion of the existence of periodic solutions of the third kind in the problem of three bodies, which has formed the basis for later researches by Poincaré and other writers. It is the purpose of this paper to present an alternative demonstration of the existence of these solutions, using certain results of von Zeipel.

26. Professor C. A. Fischer: *Functions of lines.*

The first part of this paper contains an elementary treatment of such concepts as the continuous functional, or, as it is sometimes called, function of a line, that is, curve, the linear functional and the derivative and differential of a functional. Then completely continuous transformations and Stieltjes integral equations are discussed briefly.

27. Professor I. J. Schwatt: *The expansion of a certain function.*

The author shows how the expansion of

$$(a_1 \sin^{p_1} x + a_2 \cos^{p_2} x)^n,$$

in powers of x, where p_1 and p_2 are positive integers and n is any real number, depends on the expansion

$$\sin^{p_1} x \cos^{p_2} x = \frac{1}{2^{p_1+p_2}} \sum_{\kappa=0}^{\infty} (-1)^{\kappa} \frac{x^{2\kappa+p_1}}{(2k+p_1)!} \sum_{\alpha=0}^{p_2} \binom{p_2}{\alpha} \times \sum_{\beta=0}^{p_1} (-1)^{\beta} \binom{p_1}{\beta} (p_1 + p_2 - 2\alpha - 2\beta)^{2\kappa+p_1}.$$

28. Professor I. J. Schwatt: *The summation of a family of deranged series.*

The author finds the sum of the series obtained by retaining throughout

$$S = \sum_{n=0}^{\infty} \frac{(-1)^n}{a + nh}$$

groups of p successive terms, beginning with the first term, and omitting q successive terms after each of these groups. In this case the known tests fail to disclose the conditions under which the resulting series is convergent or divergent. These conditions are determined from the sum of the series. The series is convergent for all values of p and q, except when p and q are both odd, in which case the series is divergent.

29. Professor I. J. Schwatt: *The sum of the harmonic series.*

The author has developed an expression for $Q_{n,k}$, the sum of the products of 1, 2, 3, $\cdots$, n taken k at a time, which he believes to be new.

$$Q_{n,k} = \prod_{a=1}^{n} \Big(\sum_{k_a=k+1}^{n-k+a} k_a \Big), \qquad k_a = 0,$$

$$= \frac{(-1)^k}{n!} \binom{n}{k} (n+1+k)!$$

$$\sum_{a=1}^{k} \frac{1}{(n+1+\alpha)(k+\alpha)!} \binom{k}{\alpha} \sum_{\beta=1}^{a} (-1)^{\beta} \binom{\alpha}{\beta}_{\beta}^{k+a};$$

then

$$Q_{n,n-1} = n! \sum_{k=1}^{n} \frac{1}{k} = \sum_{k=1}^{n-1} \frac{n+k}{k!} \binom{2n}{n-k-1} \sum_{a=1}^{k} (-1)^a \binom{k}{\alpha}_{a}^{n-1+k}.$$

By means of this result the sum of the harmonic series is expressed in powers of n.

R. G. D. RICHARDSON,
Secretary.

A PROPERTY OF CERTAIN FUNCTIONS WHOSE STURMIAN DEVELOPMENTS DO NOT TERMINATE *

BY O. D. KELLOGG

Let $[u_k(x)]$ be a set of continuous Sturmian functions defined on the interval (a, b), i.e., solutions of an equation of the form

$$\frac{d}{dx}[k(x)\frac{d}{dx}u(x)] + \lambda g(x)u(x) = 0, \tag{1}$$

each satisfying two linear homogeneous self-adjoint boundary conditions and corresponding to a value of λ for which this is possible. We assume that the coefficients of the differential equation have derivatives of all orders, and that $g(x)$ does not vanish in the closed interval (a, b), nor $k(x)$ on the open interval. Let $f(x)$ denote a function with derivatives of all orders, satisfying boundary conditions to be specified presently. We proceed to call attention to a property which such functions must have if their developments in series of the functions $u_k(x)$ are not to terminate.

We denote by a_k the kth generalized Fourier coefficient of f:

$$a_k = \int f u_k g dx, \tag{2}$$

where we have omitted argument x, and limits of integration, a and b. No ambiguity will result from the abbreviation. If in the integral, we replace $u_k g$ by its value obtained from the differential equation (1), and integrate by parts, we obtain

$$a_k = -\frac{1}{\lambda_k}\int (kf')' u_k dx, \tag{3}$$

where the integrated terms have been omitted on the assumption that f satisfies the same self-adjoint boundary conditions as the u_k. Under this assumption, they vanish. We define a series of functions as follows:

$$f_n = -\frac{(kf'_{n-1})'}{g}, \qquad f_0 = f. \tag{4}$$

* Presented to the Society, Feb. 25, 1922.

We now subject f to the hypothesis that it, and these derived functions, all satisfy the same boundary conditions as the u_k. It then appears that for all n,

$$a_k = \frac{1}{\lambda_k^n} \int f_n u_k g dx. \tag{5}$$

This is a simple generalization of the long familiar equation for the coefficients of a Fourier series. It is rather in the inference drawn from it, than in the generalization itself, that the interest lies. From the equation (5) we pass to an inequality. Let B_k denote the maximum of $|(b-a)g(x)u_k(x)|$, and F_n the maximum of $|f_n(x)|$. Then, evidently,

$$|a_k| \leqq B_k F_n / |\lambda_k|^n, \quad \text{or} \quad F_n \geqq (|a_k|/B_k)|\lambda_k|^n. \tag{6}$$

It follows that *unless the development of $f(x)$ terminates, F_n must, for all n greater than a determinable number N, exceed any exponential function of n, Ae^{pn},* because of the known property of the characteristic numbers λ_k of having infinity as the only limit point of their absolute values.

In other words, *if F_n is less than any such exponential function, for positive A and p and for infinitely many values of n, $f(x)$ is a homogeneous linear function of the $u_k(x)$ with constant coefficients.*

This property takes on special interest when k and g are constant, for in this case the f_n are proportional to derivatives of f. With suitable boundary conditions, it then takes the form: *if the periodic function $f(x)$ has derivatives of all orders, it is either a trigonometric polynomial, or else the maximum of the absolute value of its nth derivative exceeds any exponential function Ae^{pn} for all n from a certain one on.**

In the case of analytic functions of a complex variable, we have the result:

Either $f(z)$ is a polynomial, or else the maximum of the absolute value of its nth derivative on any circle lying entirely within its domain of analyticity exceeds any exponential function Ae^{pn} from a given n on.

HARVARD UNIVERSITY

* The result in this particular form was announced to the Society Dec. 2, 1911. See this BULLETIN, vol. 18, p. 234.

RELATING TO THE PROOF OF AN EXISTENCE THEOREM FOR A CERTAIN TYPE OF BOUNDARY VALUE PROBLEM*

BY H. T. DAVIS

1. *Introduction.* The problem of proving the existence of a solution of an ordinary, linear, homogeneous differential equation of nth order which satisfies n linear, homogeneous boundary conditions of $(n-1)$st order at p points has been attacked in various ways since it was first formulated by Sturm in 1836.† One of the most fruitful of these methods was that first employed by Bôcher,‡ and later applied by Birkhoff § and Ettlinger¶ to general self-adjoint systems of second order. This method is essentially the study of the zeros of a certain determinant of nth order by means of the properties of a system for which the existence of solutions has already been established.

The object of the present paper is to consider the problem by means of integral equations, from which is obtained a criterion of some elegance, although the generality of this method is somewhat limited by the way in which the parameter enters the integral equation.

2. *Theorems to be used.* Suppose $L(u)$ is any linear, homogeneous, ordinary differential expression of nth order and suppose the coefficient of $u^{(n)}(p_0)$ is different from zero in the interval $a \leqq x \leqq b$. Consider, then, the two systems:

$$\text{(I)} \quad \begin{cases} L(u) + \lambda u = 0, \\ U_i(u) = 0, \end{cases} \qquad \text{(II)} \quad \begin{cases} L(u) + \lambda u = 0, \\ V_i(u) = 0, \end{cases}$$

$(i = 1, 2, 3, \cdots, n)$, where U_i and V_i are linear combinations of u and its first $(n-1)$ derivatives at the two points a and b.

* Presented to the Society, April 15, 1922.

† JOURNAL DE MATHÉMATIQUES, vol. 1 (1836), pp. 106–186.

‡ COMPTES RENDUS, vol. 140 (1905), p. 928.

§ TRANSACTIONS OF THIS SOCIETY, vol. 10 (1909), pp. 259–270.

¶ TRANSACTIONS OF THIS SOCIETY, vol. 19 (1918), pp. 79–96.

We shall suppose that the existence of an infinite sequence of λ's $(\lambda_1, \lambda_2, \cdots)$ is known for system I. Our object is to show how this fact may be used to find conditions upon the coefficients of system II which will insure the existence of a sequence of λ's $(l_1, l_2, \cdots)$ for this system also.

We make use of the following theorem in the theory of integral equations:

If the Green's function of the differential expression $L(u)$ for such a set of boundary conditions as I *or* II *is taken as the kernel of an integral equation of the second kind,*

$$f(x) = \phi(x) - \lambda \int_a^b K(x, t)\phi(t)dt,$$

the Green's function of the differential expression $L(u) + \lambda u$ which corresponds to the boundary conditions I *or* II *is the resolvant function $K(x, t)$ of this integral equation.**

We recall that if

$$(1) \qquad g(x, t) = \pm \frac{\begin{vmatrix} u_1(t) & \cdots & u_1^{(n-2)}(t) & u_1(x) \\ u_2(t) & \cdots & u_2^{(n-2)}(t) & u_2(x) \\ \cdot & \cdots & \cdot & \cdot \\ u_n(t) & \cdots & u_n^{(n-2)}(t) & u_n(x) \end{vmatrix}}{2p_0(t)W(t)},$$

where $W(t)$ is the Wronskian of the fundamental set of solutions, $u_1, \cdots, u_n$, of the equation $L(u) = 0$, and their derivatives to the $(n-1)$st order, and where the plus sign refers to $x \leqq t$, and the minus sign to $x \geqq t$, then

$$G(x, t) = g(x, t) + c_1u_1(x) + \cdots + c_nu_n(x)$$

is the Green's function of $L(u)$ provided $c_1, \cdots, c_n$ are so chosen that $G(x, t)$ will satisfy the boundary conditions $U_i(u) = 0$.

We notice that $G(x, t)$ exists and is a continuous function

* Hilbert, *Grundzüge einer allgemeiner Theorie der linearen Integralgleichungen*, Teubner, 1912, p. 49. Here the theorem is proved only for the special self-adjoint case of second order, but a consideration shows that the same proof may be extended to include the case of a system of nth order.

in both arguments except when

$$F(\lambda) = \begin{vmatrix} U_1(u_1) & \cdots & U_1(u_n) \\ \cdot & \cdot & \cdot \\ U_n(u_1) & \cdots & U_n(u_n) \end{vmatrix} = 0. \tag{2}$$

3. *Interrelation of Problems* I *and* II. We shall now suppose that $\Gamma_1(x, t)$ and $\Gamma_2(x, t)$ are the Green's functions of $L(u) + \lambda u$ for boundary conditions I and II, and that $K_1(x, t)$ and $K_2(x, t)$ are the Green's functions of $L(u)$ for the same boundary conditions. Consequently Γ_1 and Γ_2 are the resolvant functions for the integral equations

$$u(x) = \lambda \int_a^b K_1(x, t)u(t)dt, \qquad u(x) = \lambda \int_a^b K_2(x, t)u(t)dt,$$

and we are led to consider the two Fredholm determinants D_1 and D_2 which belong to problems I and II.

By the well known property of these determinants,* we have:

$$\frac{dD_1}{d\lambda} = -D_1 \int_a^b \Gamma_1(x, x)dx, \qquad \frac{dD_2}{d\lambda} = -D_2 \int_a^b \Gamma_2(x, x)dx.$$

Since the zeros of $D_1(\lambda)$ and $D_2(\lambda)$ are the characteristic values $(\lambda_1, \lambda_2, \cdots)$ and $(l_1, l_2, \cdots)$ for which solutions of I and II exist, we consider the ratio D_1/D_2. Between any two characteristic values λ_i and λ_{i+1}, D_1/D_2 is a continuous function providing D_2 does not vanish in the interval. Hence its derivative must vanish in the interval:

$$\begin{aligned} \frac{d}{d\lambda}\left(\frac{D_1}{D_2}\right) &= \frac{D_2 \dfrac{dD_1}{d\lambda} - D_1 \dfrac{dD_2}{d\lambda}}{{D_2}^2} \\ &= \frac{-D_2 D_1 \displaystyle\int_a^b \Gamma_1 dx + D_1 D_2 \int_a^b \Gamma_2 dx}{{D_2}^2} \\ &= -\frac{D_1}{D_2} \int_a^b [\Gamma_1(x, x) - \Gamma_2(x, x)]dx. \end{aligned}$$

* Hilbert, loc. cit., pp. 12–13.

But suppose that it is shown that $\int_a^b(\Gamma_1 - \Gamma_2)dx \neq 0$ for $\lambda_i \leqq \lambda \leqq \lambda_{i+1}$. Our hypothesis that D_1/D_2 is a continuous function is contradicted. Consequently D_2 must vanish in the interval. This result may be stated in the following theorem.

THEOREM. *If the existence of a sequence of* λ*'s for which solutions of problem* I *exist has been established, then a sequence of l's for problem* II *alternating or coincident with the* λ*'s of problem* I *exist provided*

$$\int_a^b [\Gamma_1(x, x) - \Gamma_2(x, x)]dx \neq 0, \quad (\lambda_0 \leqq \lambda \leqq \lambda_m),$$

where Γ_1 *and* Γ_2 *are the Green's functions belonging to problems* I *and* II.

4. *Examples.* Two examples serve to illustrate the theorem:

$$u'' + \lambda u = 0,$$

$$(1) \qquad \text{I.} \quad \begin{cases} u(0) = 0, \\ u(1) = 0, \end{cases} \qquad \text{II.} \quad \begin{cases} u(0) = 0, \\ u'(1) = 0. \end{cases}$$

Specialize u_1 and u_2 by assuming $u_1(0) = 0$, $u_1'(0) = 1$; $u_2(0) = 1$, $u_2'(0) = 0$. Then

$$\int_0^1 (\Gamma_1 - \Gamma_2)dx = -\int_0^1 \frac{u_1^2(x)}{u_1(1)u_1'(1)}dx$$

which is never zero. Consequently the characteristic numbers of the two systems must always alternate with one another, a conclusion easily verified by explicitly solving the two systems.

$$(pu')' + qu + \lambda u = 0,$$

$$(2) \qquad \text{I.} \quad \begin{cases} u(a) = 0, \\ u(b) = 0, \end{cases} \qquad \text{II.} \quad \begin{cases} u(a) - u(b) = 0. \\ u'(a) - u'(b) = 0. \end{cases}$$

Again specialize the fundamental system by assuming $u_1(a) = 0$, $u_1'(a) = 1$, $u_2(a) = 1$, $u_2'(a) = 0$.

Then

$$\int_a^b (\Gamma_1 - \Gamma_2)dx = \frac{1}{D_1D_2}\int_a^b \{[u_2^2(b) - (1 + p_1)u_2(b) + p_1]u_1^2(x) - [2u_1(b)u_2(b) - (p_1 + 1)u_1(b)]u_1(x)u_2(x) + u_1^2(b)u_2^2(x)\}dx,$$

where D_1 and D_2 are the characteristic functions (2) of the two systems and $p_1 = p(a)/p(b)$. A sufficient condition, then, that the characteristic numbers of the two systems shall either alternate or coincide, is that the quadratic form in $u_1(x)$, $u_2(x)$ shall be definite. But the discriminant Δ of the form is

$$\Delta = (p_1 - 1)^2 u_1^2.$$

Consequently the form will be definite if and only if we have $p(a) = p(b)$, which is the well known condition that system II shall be self-adjoint.*

THE UNIVERSITY OF WISCONSIN

NOTE CONCERNING THE ROOTS OF AN EQUATION

BY K. P. WILLIAMS

Professors Carmichael and Mason have published the following theorem.†

All roots of the equation

$$x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_n = 0 \tag{1}$$

are, in absolute value, less than

$$\sqrt{1 + |a_1|^2 + |a_2|^2 + \cdots + |a_n|^2}. \tag{2}$$

It is apparent that this limit may be greatly in excess of the actual maximum of the absolute values of the roots. An illustration of this fact is furnished by the equation

$$x^n + x^{n-1} + \cdots + x + 1 = 0. \tag{3}$$

The theorem asserts that $\sqrt{n+1}$ is greater than the absolute value of any root. If n is large this is rather meager and inexact information, since all roots are in absolute value exactly 1, irrespective of the value of n.

* *Note on the roots of algebraic equations*, this BULLETIN, vol. 21 (1914), p. 21.

† This example is treated by Bôcher by different means in his *Leçons sur les Méthodes de Sturm*, 1917, pp. 83–91.

It is possible to modify in a very simple way the work of the authors quoted, and to obtain a theorem that in some cases gives much more exact knowledge. The work will merely be sketched in order to avoid unnecessary repetition of the work of the paper in question. It is shown in that paper that all roots of the given equation are not greater in absolute value than

$$\limsup_{m\to\infty} \sqrt[m]{C_m},$$

where

$$C_m = -\begin{vmatrix} 1 & 0 & 0 & \cdots & a_1 \\ a_1 & 1 & 0 & \cdots & a_2 \\ a_2 & a_1 & 1 & \cdots & a_3 \\ a_3 & a_2 & a_1 & \cdots & a_4 \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix}.$$

The theorem cited is then derived by applying to this determinant the theorem of Hadamard relative to a maximum value for a given determinant.

Before applying the theorem of Hadamard it is evidently possible to modify the determinant in various ways, and a new theorem will result from each modification. Let us subtract the first column from the last, the second from the first, the third from the second, etc. We have then the determinant

$$C_m = -\begin{vmatrix} 1 & 0 & \cdots & a_1 - 1 \\ a_1 - 1 & 1 & \cdots & a_2 - a_1 \\ a_2 - a_1 & a_1 - 1 & \cdots & a_3 - a_2 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_n - a_{n-1} & \cdot & \cdots & -a_n \\ -a_n & a_n - a_{n-1} & \cdots & \\ 0 & -a_n & \cdots & 0 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \end{vmatrix}$$

assuming $m > n$, the determinant being of order m. The application of the theorem of Hadamard then gives the theorem:

THEOREM. *All roots of the equation* (1) *are less in absolute*

value than the quantity

$$(4) \qquad \sqrt{1+|a_1-1|^2+|a_2-a_1|^2+\cdots+|a_n-a_{n-1}|^2+a_n^2}.$$

If we apply this theorem to equation (3) we see that all roots are less in absolute value than $\sqrt{2}$, irrespective of the value of n.

The quantity in (4) will evidently be smaller than the quantity (2) in many cases. As another illustration, consider the equation

$$(5) \qquad x^n+\frac{x^{n-1}}{\sqrt{1}}+\frac{x^{n-2}}{\sqrt{2}}+\frac{x^{n-3}}{\sqrt{3}}+\cdots+\frac{x}{\sqrt{n-1}}+\frac{1}{\sqrt{n}}=0.$$

The application of (2) gives

$$\sqrt{1+1+\frac{1}{2}+\cdots+\frac{1}{n}},$$

as a superior limit for the absolute values of the roots. Now Euler's constant tells us that this is of practically the same order as $\sqrt{\log n}$. We therefore could draw no conclusion as to whether the roots of (5) remain within a certain circle which does not change with n.

If, on the other hand, we apply (4), we have

$$\sqrt{1+\left(\frac{1}{\sqrt{1}}-\frac{1}{\sqrt{2}}\right)^2+\left(\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{3}}\right)^2+\cdots+\left(\frac{1}{\sqrt{n-1}}-\frac{1}{\sqrt{n}}\right)^2+\frac{1}{n}},$$

as the superior limit. The quantity under the radical is less than

$$\begin{aligned}1+\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{(n-1)n}+\frac{1}{n}&=1+\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\cdots\\&\quad+\left(\frac{1}{n-1}-\frac{1}{n}\right)+\frac{1}{n}=2.\end{aligned}$$

We therefore see that no root of (5) will, in absolute value, exceed $\sqrt{2}$, no matter how great n may be.

INDIANA UNIVERSITY

THE COMPLETE EXISTENTIAL THEORY OF HURWITZ'S POSTULATES FOR ABELIAN GROUPS AND FIELDS*

BY B. A. BERNSTEIN

1. *Introduction.* Hurwitz has proposed sets of postulates for abelian groups and fields.† If F', F'', F_n denote his sets for denumerable, continuous, and finite fields respectively, G', G'', G_n the corresponding sets for abelian groups, then I have proved in another paper‡ the following theorem.

THEOREM *A. Postulate-sets* F', F'', G', G'', G_n $(n > 1)$ *are each completely independent; postulate-set* F_n *is completely independent*§ *when, and only when,* n *exceeds* 2 *and is a power of a prime.*

The object of this note is to investigate postulate-set F_n further, and to prove the following theorem, which, together with Theorem *A* establishes the *complete existential theory*§ of each of Hurwitz's six postulate-sets for abelian groups and fields.

THEOREM *B. For postulate-set* F_n, *when* n *exceeds* 2 *and is not a power of a prime, there exists no system having the character* $(++++++)$, *but there exist systems having all the other characters; when* $n = 2$ *there exist no systems having the characters* $(-+++-+)$ *and* $(-+-+-+)$, *but there exist systems having all the other characters.*

2. *Hurwitz's Postulates* F_n *for Finite Fields.* For finite fields Hurwitz's postulates are as follows (K, $\oplus$, $\odot$ being undefined):

(A_1) If a, b, c, $a \oplus b$, $c \oplus b$, and $a \oplus (c \oplus b)$ belong to K, then $(a \oplus b) \oplus c = a \oplus (c \oplus b)$.

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* Presented to the Society April 8, 1922.

† W. A. Hurwitz, *Postulate-sets for abelian groups and fields*, ANNALS OF MATHEMATICS, (2), vol. 15 (1913), p. 93.

‡ *On complete independence of Hurwitz's postulates for abelian groups and fields*, ANNALS OF MATHEMATICS, (2), vol. 24 (1922).

§ See E. H. Moore, *Introduction to a form of general analysis*, New Haven Mathematical Colloquium, Yale University Press, p. 82.

(A_2) If a and b belong to K, then there is an element x of K such that $a \oplus x = b$.

(M_1) If $a, b, c, a \odot b, c \odot b$, and $a \odot (c \odot b)$ belong to K, then $(a \odot b) \odot c = a \odot (c \odot b)$.

(M_2) If a and b belong to K, and $a \oplus a \neq a$, there is an element x of K such that $a \odot x = b$.

(D) If $a, b, c, a \odot b, a \odot c, b \oplus c, (a \odot b) \oplus (a \odot c)$ belong to K, then $a \odot (b \oplus c) = (a \odot b) \oplus (a \odot c)$.

(N_n) K contains $n > 1$ elements.*

3. *Proof of Theorem B.* The proof of Theorem B is obtained with the help of the table below.†

(1) Systems 1–32 of the table have the characters $(\pm\pm\pm\pm\pm-)$ for F_n.

(2) Let $1'$–$32'$ be the systems obtained from 1–32 by substituting (a) the class of *least positive residues modulo n* for the class of integers and (b) the *least positive residue modulo n of* $a + b, ab, a - b$ for $a + b, ab$, and $a - b$ respectively, except that $a^2 + (b - 1)^2$ in 16 is left unchanged. Then when $n > 2$, systems $2'$–$32'$ will have the characters $(\pm\pm\pm\pm\pm+)$ except $(+++++)$; when $n = 2$, systems $1'$–$9'$, $11'$–$20'$, $22'$–$32'$ will be systems having all the characters $(\pm\pm\pm\pm\pm+)$ except $(-+++-+)$ and $(-+-+-+)$.

(3) When $n > 2$ and not a power of a prime there exists no field.

(4) When $n = 2$ there exists no system having the character $(-+++-+)$. For, since postulates (A_1) and (D) have to be contradicted, and (A_2) satisfied, $a \oplus b$ must be defined by the table

$\oplus$	0	1
0	1	0
1	1	0

Further, since postulates (M_1) and (M_2) have to be satisfied,

* Postulate-sets F' and F'' are obtained from set F_n by substituting for (N_n) respectively: (N') K is countably infinite, (N'') K has the cardinal number of the continuum. Sets G_n, G', G'' are obtained from F_n, F', F'' respectively by omitting (M_1), (M_2), and (D).

† This table may also be used conveniently in the proof of Theorem A.

$a \odot b$ must be defined by one or the other of the tables:

$\odot$	0	1
0	1	0
1	0	1

$\odot$	0	1
0	0	1
1	1	0

That is, we must have either

$$a \oplus b = b + 1 \pmod 2, \qquad a \odot b = a + b + 1 \pmod 2$$

or else

$$a \oplus b = b + 1 \pmod 2, \qquad a \odot b = a + b \pmod 2.$$

But for either case postulate (D) would be satisfied.

(5) That when $n = 2$ there is no system having the character $(-+-+-+)$ I have shown in the paper cited above.

This completes the proof of our theorem.

SYSTEMS HAVING THE CHARACTERS $(\pm\pm\pm\pm\pm\pm-)$ FOR F_n

No.	Character	K	$a \oplus b$	$a \odot b$
1	$(+++++-)$	Integers*	$a+b$	ab
2	$(++++--)$	"	$a+b$	$a+b$
3	$(+++-+-)$	"	$a+b$	0
4	$(++-++-)$	"	$a+b$	b
5	$(+-+++-)$	"	a	$a+b$
6	$(-++++-)$	"	b	$a+b$
7	$(+++---)$	"	$a+b$	1
8	$(++-+--)$	"	$a+b$	$b+1$
9	$(+-++--)$	"	0	$a+b$
10	$(-+++--)$	"	$a-b$	$a+b$
11	$(++---+-)$	"	$a+b+1$	$b+0/a$
12	$(+-+-+-)$	"	0	0
13	$(-++-+-)$	"	$b+1$	$a/0$
14	$(+--++-)$	"	0	b
15	$(-+-++-)$	"	b	b
16	$(--+++-)$	"	$b+\dfrac{0}{a^2+(b-1)^2}$	a
17	$(+--+--)$	"	$a+b$	$a+1$
18	$(+-+---)$	"	0	1
19	$(-++---)$	"	$b+1$	1
20	$(+--+--)$	"	0	$b+1$
21	$(-+-+--)$	"	$a-b$	$a-b$
22	$(--++--)$	"	$ab+a$	$a+b$
23	$(+---+-)$	"	1	$b+0/a$
24	$(-+--+-)$	"	$b+1$	$b+0/a$
25	$(--+-+-)$	"	$ab+a$	0
26	$(---++-)$	"	$a+1$	b
27	$(+-----)$	"	0	$a+1$
28	$(-+----)$	"	$b+1$	$a+1$
29	$(--+---)$	"	$a+1$	a
30	$(----+-)$	"	$b+0/a$	$b+0/a$
31	$(---+--)$	"	$ab+a$	$b+1$
32	$(------)$	"	$a+1$	$a+1$

* Positive, negative, and 0.

THE UNIVERSITY OF CALIFORNIA

ON KUMMER'S MEMOIR OF 1857 CONCERNING FERMAT'S LAST THEOREM*

BY H. S. VANDIVER

1. *Introduction.* In a previous paper under the same title,† the writer considered an article by Kummer,‡ and pointed out that the argument there used for proving certain results regarding the equation

$$x^\lambda + y^\lambda + z^\lambda = 0, \tag{1}$$

where x, y and z are integers and λ is an odd prime, is deficient and incorrect in several respects. Kummer attempts to prove four theorems which in my first paper were numbered I to IV. I pointed out that the proofs of Theorems I and IV are incomplete, and that the proofs of Theorems II and III are inaccurate. In the present paper additions to and modifications of Kummer's arguments will be given, by means of which the demonstrations Theorems I and IV will be completed.

2. *Proof of Theorem* I. Assume that h_1 is the first factor of the class number of the field $\Omega(\alpha)$, α being a primitive λth root of unity,

$$E_\nu(\alpha) = \prod_{k=0}^{\mu-1} e(\alpha^{\gamma^k})^{\gamma^{-2k\nu}},$$

$$e(\alpha) = \sqrt{\frac{(1-\alpha^\gamma)(1-\alpha^{-\gamma})}{(1-\alpha)(1-\alpha^{-1})}},$$

where γ is a primitive root of λ, and $\mu = (\lambda - 1)/2$; then Kummer's Theorem I may be stated as follows.

If h_1 is divisible by λ but not by λ^2, then one and only one Bernoulli number B_ν in the set B_i, $i = 1, 2, \cdots, \mu - 1$, is divisible by λ. If under this assumption we also have $h_2 \equiv 0 \pmod{\lambda}$, h_2 being the second factor of the class number of $\Omega(\alpha)$, then $E_\nu(\alpha)$ is the λth power of a unit in $\Omega(\alpha)$.

* Presented to the Society October 30, 1920.

† PROCEEDINGS OF THE NATIONAL ACADEMY, vol. 6, No. 5, pp. 266–69 (May, 1920).

‡ MATHEMATISCHE ABHANDLUNGEN OF THE BERLIN ACADEMY, 1857, pp. 41–74.

To complete Kummer's proof it is necessary to prove only the first statement in the theorem. Assume $\lambda > 5$. In another paper* the writer has given the relation

$$h_1 \equiv \prod_s \frac{\lambda(-1)^{(s\lambda^2-1)/2} B_{(s\lambda^2+1)/2}}{2^{(\lambda-3)/2}} \pmod{\lambda^2}, \tag{2}$$

$$(s = 1, 3, \cdots, \lambda - 2).$$

We shall now show that the assumption that two or more of the B's in the set B_i, $i = 1, 2, \cdots, \mu - 1$, are divisible by λ leads to the relation $h_1 \equiv 0 \pmod{\lambda^2}$ which will yield the proof desired, as it is known† that if $h_1 \equiv 0 \pmod{\lambda}$ then at least one of the B's is divisible by λ.

Assume that $B_a \equiv B_b \equiv 0 \pmod{\lambda}$, where a and b are each included in the set $1, 2, \cdots, \mu - 1$. Kummer‡ has shown that

$$\frac{B_c}{c} \equiv (-1)^{k\mu} \frac{B_{c+k\mu}}{c + k\mu} \pmod{\lambda},$$

where k is an integer and c is not a multiple of μ, and where $\lambda > 3$. This gives

$$(-1)^{s\mu(\lambda+1)} \frac{B_{(s\lambda^2+1)/2}}{(s\lambda^2+1)/2} \equiv \frac{B_{(s+1)/2}}{(s+1)/2} \pmod{\lambda},$$

and if $(s + 1)/2 = a$, then s is included in the set $1, 3, 5, \cdots \lambda - 2$, and the above congruence gives

$$B_{(s\lambda^2+1)/2} \equiv 0 \pmod{\lambda}.$$

Similarly

$$B_{(s_1\lambda^2+1)/2} \equiv 0 \pmod{\lambda},$$

where $(s_1 + 1)/2 = b$. Applying these relations to (2), we obtain $h_1 \equiv 0 \pmod{\lambda^2}$, and Theorem I is proved, as the remainder of the proof as set forth by Kummer, is, in my opinion, rigorous.

3. *Proof of Theorem* IV. We proceed to Kummer's fourth theorem:

THEOREM IV. *If h_1 is divisible by λ, but not by λ^2, and P is an ideal in $\Omega(\alpha)$, such that P^λ is the principal ideal $(F(\alpha))$,*

* This BULLETIN, vol. 25, p. 460, relation (8) for $a = 2$.

† Vandiver, ibid., p. 461.

‡ CRELLE, vol. 41, p. 368.

$B_\nu \equiv 0 \pmod{\lambda}$, $\nu < \mu$, then P is or is not a principal ideal according as

$$\frac{d_0^{\lambda-2\nu} \log F(e^v)}{dv^{\lambda-2\nu}} \equiv 0 \text{ or } \not\equiv 0 \pmod{\lambda}.$$

In this statement, the notation $d_0^{\lambda-2\nu}$ means that the function $F(e^v)$ is to be differentiated $\lambda - 2\nu$ times with respect to v, and zero substituted for v in the result. The letter e denotes the Napierian base.

In my first note two criticisms of Kummer's proof of this theorem were made. I shall here modify his argument so as to dispose of the difficulties in question, and consequently complete the demonstration. The first of my criticisms referred to formula (A) on page 53 of Kummer's memoir. The number $\psi_r(\alpha)$ which appears there is defined in the eleventh line from the bottom of the page as the product of certain ideal factors, but this decomposition is proved to hold only for the case where $\psi_r(\alpha)$ contains ideals of degree not higher than the first. In another paper,* Kummer gives the corresponding formula for the generalized function $\psi_r(\alpha)$ which contains ideals of higher degree, as follows:

$$(3) \quad \text{ind } E_n(\alpha) \equiv \frac{\gamma^{2n} - 1}{2(1 + r^{\lambda-2n} - (r+1)^{\lambda-2n})} \cdot \frac{d_0^{\lambda-2n} \log \psi_r(e^v)}{dv^{\lambda-2n}} \pmod{\lambda},$$

where r is an integer, $1 < r < \lambda - 1$, $E_n(\alpha)$ and γ are defined as before, and ind $(E_n(\alpha))$ is i in the relation

$$(E_n(\alpha))^{(q^t-1)/\lambda} \equiv \alpha^i \pmod{\mathfrak{P}},$$

$\mathfrak{P}$ being an ideal prime factor of the odd prime q, and t the exponent to which q belongs modulo λ. Further

$$\psi_r(\alpha) = \alpha^{-(r+1)h + \text{ind}(g^h+1)},$$

where g is a primitive root of $\mathfrak{P}$ such that $g^{(q^t-1)/\lambda} \equiv \alpha \pmod{\mathfrak{P}}$, h ranges over the integers $0, 1, 2, \cdots, q^t - 2$, excepting $(q^t - 1)/2$, ind $(g^h + 1)$ being defined as i in the relation

* CRELLE, vol. 44, p. 125.

$(g^h + 1) \equiv g^i \pmod{\mathfrak{P}}$. Although Kummer in defining q did not state that it was odd, his work is subject to that restriction, since his function $\psi_r(\alpha)$, as defined by him, has no meaning for $q = 2$, since in that case $(q^t - 1)/2$ is not integral. If, however, we take the function as defined by H. H. Mitchell * for the case $q = 2$, namely

$$\psi_r(\alpha) = \alpha^{-(r+1)h+\mathrm{ind}(g^h+1)}$$

where h ranges over the integers $1, 2, \cdots, 2^t - 2$, the formula (3) will also hold for this case, as Kummer's argument can be used without change except that it is necessary to note in proving formulas such as the following (loc. cit., middle of page 125)

$$(g^i - 1)(g^i - g^\lambda)(g^i - g^{2\lambda}) \cdots (g^i - g^{(\nu-1)\lambda}) \equiv 1 - g^{\nu i} \pmod{\mathfrak{P}},$$

that since $1 \equiv -1 \pmod{q}$ for $q = 2$,

$$g^{\nu i} - 1 \equiv 1 - g^{\nu i} \pmod{p}.$$

Kummer (loc. cit., page 120) gives the decomposition of the general $\psi_r(\alpha)$ into prime ideal factors. Instead of using this decomposition we shall examine the form of it given by Mitchell † and express the factorization of $\psi_r(\alpha)$, t arbitrary, in a form analogous to that given by Kummer for the case $t = 1$. Mitchell ‡ considers the Galois field of order q^t, where q is any prime, and $q^t - 1 = \lambda\nu$. Let the elements of this field other than zero each be represented as a power of a primitive root, and σ_i denote any element whose index is congruent to i, modulo λ. The symbol m_i^j stands for the number of solutions of the relation $1 + \sigma_i = \sigma_j$ in the field. He then defines the function (page 167, relation 4),

$$\psi_{-a,-b}(\alpha) = \sum_{i,j} m_i^j \alpha^{-bi+(a+b)j},$$

where i and j each range over the integers $0, 1, \cdots, \lambda - 1$, and $a \not\equiv 0$, $b \not\equiv 0$, $a + b \not\equiv 0 \pmod{\lambda}$. For $b = -1$, $a = -r$,

* TRANSACTIONS OF THIS SOCIETY, vol. 17 (1916), p. 165.

† Loc. cit., p. 168.

‡ Loc. cit., p. 166, § 2.

this becomes

$$\psi_{r,1}(\alpha) = \sum_{i,\ j}^{\ \ i\ \ \ j} m_i \alpha^{-(r+1)j+i}.$$

Since there are $m_j = m_i$ numbers $h \equiv j$ and ind $(g^h + 1) \equiv i \pmod{\lambda}$, the preceding relation may be written in the form

$$\psi_r(\alpha) = \Sigma \alpha^{-(r+1)h+\text{ind}(g^h+1)}.$$

Now apply Mitchell's first theorem (loc. cit., page 173) to the particular function $\psi_r(\alpha)$. We have in this case $t = t_1$, where q belongs to the exponent t_1, modulo λ. We conclude that the number of times the ideal $\mathfrak{P}_i$, where this symbol designates the ideal obtained from $\mathfrak{P}$ by the substitution (α^k/α), $ki \equiv 1 \pmod{\lambda}$, is equal to the number of the expressions $|-riq^{t-j}|+|-iq^{t-j}|$, $j = 0, 1, \cdots, t-1$, whose values exceed λ, the integer i assuming $(\lambda - 1)/t$ values prime to λ such that the quotient of no two of them is congruent modulo λ to a power of q, and the symbol $|x|$ denoting the least positive residue of x, modulo λ. Now if we select the integers l in the set $1, 2, \cdots, \lambda - 1$, which satisfy the relation $|-rl| + |-l| > \lambda$, we may show from what precedes that

$$\prod_c \mathfrak{P}_c = \psi_r(\alpha), \tag{4}$$

where c ranges over the integers which satisfy $cl \equiv 1 \pmod{\lambda}$. For, the integers $1, 2, \cdots, \lambda - 1$, are congruent modulo λ to the integers

$$\begin{array}{l} i_1,\ i_1q,\ i_1q^2,\ \cdots,\ i_1q^{t-1}, \\ i_2,\ i_2q,\ i_2q^2,\ \cdots,\ i_2q^{t-1}, \\ \cdot\quad\cdot\quad\cdot \\ \cdot\quad\cdot\quad\cdot \\ i_s,\ i_sq,\ i_sq^2,\ \cdots,\ i_sq^{t-1}, \end{array}$$

in some order, where $s = (\lambda - 1)/t$, and $i_1, i_2, \cdots, i_s$ are the integers less than λ such that the quotient of no two of them is congruent, modulo λ, to a power of q, since this definition of the i's shows that no two elements of the above array are congruent modulo λ. According to the first conclusion regarding factors of $\psi_r(\alpha)$, we may write

$$\psi_r(\alpha) = \prod_a \mathfrak{P}_a^{d_a}, \tag{5}$$

where a ranges over the integers $i_1, i_2, \cdots, i_s$, and d_a is the number of expressions

$$|-ri_aq^{t-j}|+|-i_aq^{t-j}|, \quad (j=0, 1, 2, \cdots, t-1),$$

whose values exceed λ. Since $\mathfrak{P}(\alpha)=\mathfrak{P}(\alpha^{\sigma_i})$, a an integer, the ideals $\mathfrak{P}(i_nq)$, $\mathfrak{P}(i_nq^2)$, $\cdots$, $\mathfrak{P}(i_nq^{t-1})$, where $\mathfrak{P}(\alpha^f)=\mathfrak{P}(f)$ are all equal, and the decomposition (5) is identical with the the decomposition (4), which was to be shown. Using Kummer's notation, the relation

$$|-ri_eq^{t-j}|+|-i_eq^{t-j}|>\lambda$$

may be written in the form

$$\gamma_{\mu-h}+\gamma_{\mu-h+\mathrm{ind}\,r}>\lambda,$$

where γ_b is the least positive integer satisfying $\gamma_b \equiv \gamma^b \pmod{\lambda}$ and ind r is defined by $\gamma^{\mathrm{ind}\,r} \equiv r \pmod{\lambda}$. In view of the above we may now use the relation*

$$\psi_r(\alpha)^{H_1\lambda} = \pm \alpha^d \prod_h F(\alpha^{\gamma^h}) \tag{5a}$$

where $H_1\lambda$ is the class number of $\Omega(\alpha)$, H_1 prime to λ.

Now, as pointed out in my first article, Kummer employs (page 54), without proof or reference the relation

$$\frac{d_0^{m\lambda^r}\log\varphi(e^v)}{dv^{m\lambda^r}} \equiv \frac{d_0^{m\lambda^r}\log\varphi_1(e^v)}{dv^{m\lambda^r}} \pmod{\lambda^{r+1}},$$

m not being a multiple of $\lambda-1$, and $\varphi(\alpha)\equiv\varphi_1(\alpha) \pmod{\lambda^{r+1}}$. We shall prove a special case of this relation and use of it will enable us to complete the proof of Theorem IV. Assume that $\varphi(x)$ and $\varphi_1(x)$ are two integral algebraic functions of x with rational integral coefficients, such that $\varphi(1)=\varphi_1(1)$ and $\varphi(\alpha)=\varphi_1(\alpha)$. Further let $\varphi(\alpha)$ be prime to λ, whence it follows that $\varphi(1)\not\equiv 0 \pmod{\lambda}$. Under these conditions, it will be shown that

$$\frac{d_0^{m\lambda}\log\varphi(e^v)}{dv^{m\lambda}} \equiv \frac{d_0^{m\lambda}\log\varphi_1(e^v)}{dv^{m\lambda}} \pmod{\lambda^2}. \tag{6}$$

* Kummer gives this relation in a form not containing the factor α^d, but this appears to be an error. Compare the reasoning in the writer's paper, ANNALS OF MATHEMATICS, (2), vol. 21, p. 74, on the determination of $e(\alpha)$ in relation (6). This does not affect Kummer's later arguments, however.

We have

$$\varphi(e^v) = \varphi_1(e^v) + VW_1$$

where $V = (e^{vp} - 1)/(e^v - 1)$ and W_1 is an integral function of e^v. If we divide VW_1 by $e^{vp} - 1$, we have a remainder which must be of the form cV where c is an integer. We then write

$$\varphi(e^v) = \varphi_1(e^v) + W(e^{vp} - 1) + cV.$$

Putting $v = 0$, we have

$$\varphi(1) = \varphi_1(1) + cp,$$

and by hypothesis $\varphi(1) = \varphi_1(1)$, so that $c = 0$. Hence

$$\varphi(e^v) = \varphi_1(e^v) + W(e^{vp} - 1)$$

and

$$\frac{\varphi(e^v)}{\varphi_1(e^v)} = 1 + \frac{W(e^{vp} - 1)}{\varphi_1(e^v)} = U, \text{ say.}$$

We then have

$$(7) \quad \frac{d^{m\lambda} \log \varphi(e^v)}{dv^{m\lambda}} - \frac{d^{m\lambda} \log \varphi_1(e^v)}{dv^{m\lambda}}$$
$$= U^{-1}\frac{d^{m\lambda}U}{dv^{m\lambda}} + (m\lambda - 1)\frac{d^{m\lambda-1}U}{dv^{m\lambda-1}} \cdot \frac{d(U^{-1})}{dv}$$
$$+ \cdots + \frac{dU}{dv} \cdot \frac{d^{m\lambda-1}(U^{-1})}{dv^{m\lambda-1}}.$$

Now every derivative of U is divisible by λ, if zero is substituted for v. Hence every term of the right-hand member of the above relation consists of integers or fractions whose denominators are prime to λ (since $\varphi_1(1) \not\equiv 0 \pmod{\lambda}$) and whose numerators are divisible by λ^2, excepting possibly the first term. To examine this term, we write $X = W/\varphi_1(e^v)$, and, therefore

$$\frac{d^{m\lambda}U}{dv^{m\lambda}} = (e^{vp} - 1)\frac{d^{m\lambda}X}{dv^{m\lambda}} + m\lambda\frac{d(e^{v\lambda} - 1)}{dv} \cdot \frac{d^{m\lambda-1}X}{dv^{m\lambda-1}}$$
$$+ \cdots + X\frac{d^{m\lambda}(e^{v\lambda} - 1)}{dv^{m\lambda}}.$$

Since

$$\frac{d_0{}^k(e^{v\lambda} - 1)}{dv^k} \equiv 0 \pmod{\lambda^2},$$

for $k > 1$, it follows from these relations that

$$\frac{d_0{}^{m\lambda}U}{dv^{m\lambda}} \equiv 0 \pmod{\lambda^2};$$

and by using this with (7) the congruence (6) is obtained.

From (5a) we have

$$\psi_r(\alpha)^{H_1\lambda(\lambda-1)} = \alpha^{d(\lambda-1)}\prod_h F(\alpha^{\gamma^h})^{\lambda-1}, \tag{5b}$$

and we may write $F(\alpha^{\gamma^h})^{\lambda-1} = 1 + \omega(1-\alpha) = G(\alpha^{\gamma^h})$,

where ω is an integer in $\Omega(\alpha)$, since if

$$F(\alpha^{\gamma^h}) = a + \omega_1(1-\alpha),$$

where a is a rational integer, then

$$\begin{aligned} F(\alpha^{\gamma^h})^{\lambda-1} &= a^{\lambda-1} + \omega_2(1-\alpha) \\ &= 1 + \lambda k + \omega_2(1-\alpha) \\ &= 1 + \omega(1-\alpha), \end{aligned}$$

after using the known relation $\lambda = \epsilon(1-\alpha)^{\lambda-1}$ where ϵ is a unit in $\Omega(\alpha)$. The relation (5b) then becomes

$$\psi_r(\alpha)^{H_1\lambda(\lambda-1)} = \alpha^{d(\lambda-1)}\prod_h G(\alpha^{\gamma^h}), \tag{5c}$$

whence

$$\psi_r(1)^{H_1\lambda(\lambda-1)} = \prod_h G(1) = 1,$$

Since it is known that $\psi_r(1) = 1$. Hence we can apply (6) to (5c), and proceeding as Kummer did on pages 54–57 of his article, we obtain

$$\text{ind } E_\nu(\alpha) \equiv -\frac{(-1)^{\lambda+\mu}(\gamma^{2\nu}-1)B_{\nu\lambda-\mu}}{2H_1\lambda} \cdot \frac{d_0{}^{\lambda-2\nu}lG(e^v)}{dv^{\lambda-2\nu}},$$

modulo λ. By definition of G we also have

$$\frac{d_0{}^{\lambda-2\nu}lG(e^v)}{dv^{\lambda-2\nu}} \equiv -\frac{d_0{}^{\lambda-2\nu}lF(e^v)}{dv^{\lambda-2\nu}} \pmod{\lambda},$$

and the last two relations give the congruence obtained by Kummer (loc. cit., page 57). The remainder of his argument, up to page 61, appears to be correct, if we use the Dedekind definition of ideal. This establishes the theorem which I have designated IV.

CORNELL UNIVERSITY

HAHN'S REELLE FUNKTIONEN

Theorie der reellen Funktionen. By Hans Hahn. Vol. I. Berlin, Julius Springer, 1921. vii + 600 pp.

A preliminary indication of the contents of a book may sometimes be conveyed by a statement as to what it does not contain. Such information is supplied by the preface to the present volume, which states that a second volume, completing the work, is to present the theory of integration and differentiation, the analytic representation of arbitrary functions, and Fourier's series. A lower inequality for the order of ideas involved is given by another statement in the preface, that although the principal facts of the general theory of sets and the theory of real numbers are summarized in an introduction for convenience of reference, no systematic development of these theories is attempted, and the difficult questions which gather around their foundations are not touched upon at all.

The prospective reader will be further enlightened by a glance at the author-index at the back of the book. There he will find thirty-eight references to Lebesgue, thirty-three to W. H. Young, and twenty-four each to Baire and Hausdorff, while, at the other end of the scale, Heine is mentioned four times, Cauchy three times, Dirichlet twice, and Riemann once.

The volume is concerned, then, with those investigations of the last few decades which have had the specific purpose of throwing the fullest and most searching light on the fundamental concepts of function and limit. In a larger sense, the working out of these concepts may be regarded as the supreme achievement of research in mathematical analysis during the past ninety or a hundred years. The various special theories, while offering more or less that is of value in their own particular conclusions, have derived an added significance from their relation to the central theme, the bringing of the mysteries of eighteenth-century and early nineteenth-century mathematics within the domain of assured knowledge and common-sense. Our files of journals doubtless contain thousands of pages which will receive scant attention from generations to come, but if this generalization of an existence theorem, or that simplification of a convergence proof, has contributed in its day to the stirring of ideas through which understanding emerges, it has done its part. From this point of view, the present work represents a somewhat comprehensive version, rather than a fragment, of what has been learned in a century of analysis. It is a summary too general in its terms, too abstract and refined, to be appreciated by limited human intelligence without a background of experience and illustration, which the reader must bring to its study; but it is an account possessing a certain symmetry and completeness of its own.

It goes without saying that a book which opens with the words "We begin with a brief survey of the simplest facts in the theory of sets," which presents the Wohlordnungssatz on page 25, and which in six hundred pages does not reach the subject of integration or differentiation, is not designed

as an introductory text. An adequate appraisal of it as a work of science would require the insight of a critic much more widely and deeply read in the literature of the field than the present reviewer. His naïve impressions on turning its pages may, however, be of some value to other readers of similarly unspecialized qualifications.

As has been mentioned already, an introductory chapter summarizes the facts that are to be used subsequently with regard to sets in general, the transfinite cardinal and ordinal numbers, the real number system, and the various types of limits associated with sequences and sets of real numbers. There is no attempt to lay down a complete system of definitions and postulates. There is no definition of an infinite set, for example, and an irrational number is defined merely as a number that is not rational. The principle of choice is accepted with a simple statement that criticism of its logical basis need not be taken into account for the purpose in hand. It is used, or Zermelo's theorem based on it is used, without further question, when occasion demands, and at least once when it is not necessary, in the proof of the theorem that two sets are equivalent if each is equivalent to a part of the other. Nevertheless, one must not take too seriously the author's modest statement that the treatment is not "systematic." It will be an unusual reader whose knowledge of the subject is not materially improved both in range and in precision by a perusal of the chapter.

For the purposes of a study which is ultimately to be concerned largely with derivatives and derived numbers, it is convenient to permit functions to take on infinite values, and the real number system is formally extended in the present treatment by the adjunction of the two numbers $+\infty$ and $-\infty$. The careful working out of the idea shows clearly the difficulties with which any such definition has to contend. It is a matter of choice whether infinity shall be regarded as a number or not, but no definition can make it a number *like other numbers*. Nearly a page is devoted to an enumeration of the operations which may and those which may not be performed with the two infinite numbers. But complications are not thereby finally disposed of. For example, the operation $(+\infty) - (+\infty)$ is in general not defined (p. 28). But when the oscillation (Schwankung) of a function is defined, in a later chapter (pp. 190–191), it is stated that, for this particular purpose, $(+\infty) - (+\infty)$ shall be regarded as having the value 0. And yet, in a subsequent discussion of the same concept (pp. 214–215), it is found necessary to regard infinite values as exceptional again.

The limit of a sequence of numbers is so defined as to admit the possibility of infinite limits. "A sequence which has a limit is called convergent; in particular, if the limit is finite, the sequence is said to be properly convergent; a sequence that does not converge is called oscillating" (p. 32). The same convention is maintained in the definition of continuity (pp. 122–123), and even in those of uniform continuity (p. 131) and uniform convergence (pp. 246–247). On the other hand, when the limit of a sequence of points is introduced, while the definition that recognizes infinite limits is mentioned, it is not the one adopted for systematic use (pp. 56–57). The sequence of numbers 1, 2, 3, . . . , has the limit $+\infty$, while the

corresponding sequence of points on the axis of reals (with the customary definition of distance) has no limit.

In calling attention to these contrasts, the reviewer does not wish to deprecate the introduction of infinite numbers. On the contrary, he regards the author's careful treatment of them, in a form for convenient reference, as a particularly valuable service to students of analysis. But it must be recognized that their use is at best the choice of the lesser of two considerable evils. The difficulties are in the nature of things, and are not lightly to be smoothed away.

Chapter I, on point sets, opens with the postulates for distance in a general metric space. The rest of the treatise is similarly general in scope, with occasional specific consideration of relations in euclidean space of one or more dimensions. This does not mean, however, that its aspect is uniformly forbidding to the general reader. On the contrary, it is singularly easy to dip in almost anywhere and gain at least a partial appreciation of what is going on. For example, one reads on page 86: "A part $\mathfrak{A}'$ of an arbitrary set $\mathfrak{A}$ is called a component of $\mathfrak{A}$, if $\mathfrak{A}'$ is connected, and every connected part of $\mathfrak{A}$ which has a point in common with $\mathfrak{A}'$ is a part of $\mathfrak{A}'$." The general intent of this is fairly obvious at a glance, although a precise understanding of it naturally involves a knowledge of the technical meaning of the terms involved; and if the reader turns back a few pages and finds that "A set $\mathfrak{A}$ is said to be connected, if it is not the sum of two non-vacuous parts, each closed on $\mathfrak{A}$," he may come to the conclusion that there is more in the definition first quoted than he thought. Of course it occasionally happens that one misses the point of a statement entirely, unless one has the exact meaning of the terminology in mind, as, in the latter definition, the distinction between *sum* (Summe) and *union* (Vereinigung) of two sets (p. 2). For the most part, however, be it repeated, a fair understanding of the text is surprisingly accessible even to the casual reader, such is the excellence of the presentation.

Chapter II deals with the concept of function, upper and lower bounds and limits of a function on a point set or at a point, continuity, and semi-continuity. It is to be remembered that the terms point and point set refer to elements and sets of elements in any kind of "space" which satisfies the necessary postulates. Mention has already been made of the admission of infinite values for a function; the definitions are of course framed for functions defined on an arbitrary point set, so that, for example, "at an isolated point of $\mathfrak{A}$ every function f is continuous on $\mathfrak{A}$" (p. 125), whether its value there is finite or infinite. Attention may be called to the sections on the continuous extension of a continuous function beyond its original domain of definition, and on correspondences between point sets, such correspondence being treated as a generalization of the function idea.

Chapter III is concerned with discontinuous functions, the oscillation functions associated with them, the distribution of points of discontinuity, and pointwise discontinuous functions in particular. The next chapter treats of sequences of functions: of continuous convergence at a point, uniform convergence at a point or over a point set, and related concepts. The fifth chapter gives a very systematic and satisfying account of the

Baire classification of functions. The last three chapters, on absolutely additive functions of sets, functions of limited variation, and measurable functions, cover a considerable part of what is commonly regarded as the theory of integration, though the process of integration itself is not discussed. The treatment of the measure of point sets in the sixth chapter is an especially interesting example of the method of postulates, several groups of theorems being developed in succession as successive restrictions are placed on the notion of measure.

The material appearance of the book is indicative of unremitting effort under adverse conditions. The paper, though fairly white and clear, is almost of the consistency of blotting-paper; the typography, on the other hand, is so excellent that a fairly attentive turning of the pages has disclosed only some half-dozen trifling misprints.

The exposition is remarkably clear and systematic throughout. If sometimes rather diffuse for a mere presentation of the facts, it is the more convenient for purposes of reference. The book is written neither as an exhibition of the author's learning, nor as a memorial to an abstractly conceived body of truth, orderly and symmetric as it is; it is written for the reader's information, by a man who anticipates the reader's difficulties and provides against them with great faithfulness and skill. If it be true that genius is nothing more than an infinite capacity for taking pains, this is unquestionably a work of genius; if it is possible to imagine a certain quality of inspiration which seldom thrills the reader of these pages, it is perhaps only because passages which would be regarded as evidence of inspiration in the case of a less eminently qualified writer are here accepted as a mere matter of course.

DUNHAM JACKSON.

SHORTER NOTICES

The Mathematical Theory of Electricity and Magnetism. By J. H. Jeans. Fourth Edition. Cambridge, University Press, 1920. vi + 627 pp.

Jeans's text, first printed while he was at Princeton fifteen years ago, has ever since been one of the very few standard texts in its particular field. Some substantial rearrangements and additions were made in the second edition (1911), and more are now found in the fourth, especially with reference to relativity. Nothing is offered on quantum theory or on the modern electrical theory of matter. Though there is some account of electron theory (of the Lorentz-Drude type), the book is essentially an introduction to the Maxwell theory with the additions of those parts of electromagnetic theory most naturally flowing out of Maxwell's point of view. The reason might not be far to seek: the work is, as it states, on the mathematical theory of electricity and magnetism, written by one who revels in the mathematical treatment of nature's problems, and the most recent developments of physical hypothesis relative to the electrical and quantum theories of matter are hardly yet emerging into a mathematical theory of sufficient completeness and elegance to rejoice the heart.

I have read Jeans's book more than once with classes of students who had had a good training in calculus and in the elements of electricity and magnetism; we have found in the text and problems excellent and practicable material for a full year's course with plenty to spare. It is good for young people who plan to be physicists or higher engineers to study the text and to work the problems. A facility is thereby gained in the application of analytic processes to nature. Better headway could be made if students had a better training in analytic mechanics. Greater interest could be more readily maintained if the book were more physical, if more problems were stated in terms of volts, ampères, ohms, etc., and if a greater emphasis were laid on checking the results against experiment. A student may do excellent work upon this text without gaining at all in his sense of contact with nature,—this is often the case with mathematical physics.

Pedagogically the text is very uneven, now easy, now difficult; some of the exercises are almost trivial, others exceed the ability of the best students. The teacher must be wary. Inasmuch as we are offered ample material for two years of study and in view of the real teaching difficulty of developing in a student at one time both the physical and the analytic sense, I incline to the belief that the text would be greatly improved by a major rearrangement of the material, including the exercises, so that first one might have the physical developments and those exercises in which the difficulties are the physical statement and the analysis of the problem, and second the mathematical elaboration and the exercises in which the tricks of mathematical analysis offer the obstacles to be overcome. It is not easy for the teacher to work out this assortment from the text as now composed. However, I do not know whither I should turn for a book that would work out any better for my course on the mathematical theory of electricity and magnetism.

E. B. Wilson.

Problems of Cosmogony and Stellar Dynamics. By J. H. Jeans. Adams Prize Essay for 1917. Cambridge, University Press, 1919. vii + 293 pp.

As a general cultural study, astronomy is comparable to poetry in its stimulation of the imagination. It appeals to the widest interests, the most diverse individuals. And of all astronomy the most fascinating is cosmogony—the origin, evolution, and destiny of our material universe. The arrangement pursued by Jeans in his Adams Prize Essay was apparently adopted with the aim of permitting an intelligent reader to carry the general drift of his theory without having to wade through deterring mathematical difficulties. Recondite mathematics there is in plenty, but always with a pleasant prelude and with an effective summary.

Briefly, Jeans's conception of stellar evolution follows Russell and his notion of the cosmos as a whole is the "island universe" theory. Our galactic system is comparable in mass with the largest spiral nebula. The spiral nebulæ are not part of our galactic system, which is a star cluster larger than other star clusters as the Andromeda nebula is larger than other spirals. The development of the individual stars is from cold diffuse masses of cosmic dust up through the red to the white or even blue type, and then back to the dense dying reds. The theories of figures of equilibrium changing slowly toward a limiting unstable state with subsequent rapid cataclysmic development are marshalled in support of this general conception of the world. That some differ with the author even in the dynamical theory, to say nothing of the general conclusions, may be seen by reference to Macmillan's long analytical review in the ASTROPHYSICAL JOURNAL. For a discussion of the tenability of the island-universe theory, one may read the symposium of Curtis and Shapley before the National Academy in 1920, printed by the National Research Council.

As I am merely one of the many who have read avidly the writings of mathematical and physical astronomers since the time I could first read at all, I have but small right to any opinion on cosmogony, but may I venture a word in admiration of the courage of those who work in this difficult field where there are rare clews and, it sometimes seems, no proofs. There are only seven stars (of course, binaries) of which the masses are well determined, and these lie between 0.7 and 3.4 if the sun's mass be unity; yet it is generally stated that the masses of all the stars are of approximately the same magnitude. This is a courageous conclusion. May it not be that the eight (our sun included) are not a fair random sample of the billion and a half? And even if you have a random sample of eight out of a lot of a billion and a half what is the probable departure of the average of any particular physical property of the eight from that of the lot? Jeans is aware of such troubles; he is tentative, not dogmatic. (Since this review was first written, a discovery has been reported of a double star system of which the mass is many times that of the sun.)

The essay contains a review of earlier work by Roche, Darwin, Poincaré, expanded with many individual contributions which the author has been making in recent years in friendly competition with Eddington. Those who read the work will feel the thrill of Keats "On First Looking into Chapman's Homer."

E. B. WILSON.

Storia della Geometria Descrittiva dalle Origini sino ai Giorni Nostri. By Gino Loria. Milano, Ulrico Hoepli, 1921. xxiv + 584 pp.

Descriptive geometry has not been neglected in the histories of mathematics and considerable historical information is to be found in some of the text-books. We have also Poudra's *Histoire de la Perspective Ancienne et Moderne* (1864), Cremona's *Sulla Storia della Prospettiva Antica e Moderna* (1865), and Obenrauch's *Geschichte der darstellenden und projectiven Geometrie* (1897). Nevertheless, there is a place for a book of the present day devoted exclusively to the evolution of descriptive geometry,—especially when it is written by one so well qualified for the task as is Professor Loria.

As the author considers descriptive geometry a branch of pure mathematics, he treats briefly the history before the time of Monge. This is done in the first three chapters, which contain one-sixth of the total number of pages of the text. The next two chapters, which add almost another sixth to the book, deal with Monge and his contemporaries and disciples. The treatment of the material in the first hundred and twenty pages of the text is not new to the reader who is familiar with Professor Loria's article (XXV) in the fourth volume of Cantor's *Vorlesungen über Geschichte der Mathematik.*

The further development of descriptive geometry until about 1885 occupies seven chapters, almost one-half of the text. Six of these follow the history in different countries. In the chapters on Italy, France, Germany, German Switzerland, and Austria-Hungary, a short preliminary statement of conditions is followed by separate sections devoted to critical studies of the writings of the principal men. These chapters are concluded by sections treating the less important writers. The other countries are disposed of in one short chapter. The twelfth chapter deals with a special subject, axonometry, and traces its development from 1820 to the present.

In the last chapter of the book (the development from 1885 onward) the separate sections are entitled: (1) general considerations, (2) old and new methods of representation, (3) plane and space curves, (4) surfaces, (5) illumination of surfaces, (6) descriptive geometry of ruled, of higher, and of non-euclidean spaces, and (7) teaching and history of the subject. In a three-page conclusion the author sets forth the lines along which he predicts that the descriptive geometry of the future can and will develop.

The twelve-page index of proper names is a valuable addition to the text, although there are some misprints. It contains about seven hundred fifty names and one hundred thirty references to Monge. In footnotes scattered throughout the text there are more than a thousand references.

The author has gathered together a large collection of facts. He has arranged them methodically and he has given his sources. And he has done something more, for he has shown an appreciation of relative values by stressing the more important things and subordinating the minor details, he has displayed an insight into the vital relation between his own subject and other branches of learning, and he has created an atmosphere that is likely to inspire the reader with his own enthusiasm for descriptive geometry.

E. B. Cowley.

Plane Algebraic Curves. By Harold Hilton, M.A., D.Sc. Oxford, The Clarendon Press, 1920. 16 + 388 pp.

This is the first treatise on plane algebraic curves to appear in English since Salmon's famous treatise which was published over forty years ago. During the last four decades many new theorems have appeared in the various mathematical journals and thus it is fitting that a new English treatise should be written.

In dealing with such an extensive subject it is of course impossible to include all the known material, and one must expect to find omissions of certain topics. The most striking omission in this text is the modern algebraic-geometric development as set forth by Castelnuovo, Severi, Segre and others. In fact, the whole topic of geometric transformations and the derivation of properties of algebraic curves by means of these transformations has been omitted except for two cases of the quadratic transformations. Thus one is impressed by the fact that this treatise follows to a great extent the same general course as Salmon's, except that topics are more fully discussed and are brought up to date. However, many excellent collections of exercises are scattered throughout the book, and these alone are well worth the price of the text. The great majority of the results have been derived by algebraic methods. In many places the work could be shortened if synthetic methods were used.

The main topics discussed are: coordinate systems, projection, singular points, curve tracing, tangential equations and polar reciprocals, foci, superlinear branches, polar curves, Hessian, Steinerian, Cayleyan, Jacobian of three curves, Plücker's numbers, deficiency, higher singularities, two types of quadratic transformations, unicursal curves, derived curves, intersection of curves, unicursal cubics, non-singular cubics, cubics as Jacobians, use of parameter for non-singular cubics, unicursal quartics, quartics of deficiency one and two, non-singular quartics, ovals and circuits, corresponding ranges and pencils.

This book should be found in every mathematical library, for the topics discussed are most admirably treated. As a text for a first course it is superior to anything that has appeared as yet in any language because of the excellent collection of exercises.

F. M. Morgan.

Vector Calculus. By Durgaprasanna Bhattacharyya. Calcutta, University of Calcutta, 1920. 90 pp.

In order to present the essential features of vector analysis for use in mathematical physics, the author develops the differential and integral calculus of vectors and functions of vectors directly. He scorns the use of coordinates very properly with much the feeling of Tait that one "should not violate the spirit of the Order." However, he does not overlook the great practical advantage that accrues from the study of ordinary vectors.

The author defines line, surface, and volume integrals of vectors and of vector functions. He then defines gradient of a scalar function F substantially as the vector F in the expression $d\rho \cdot F$. After considering the linear vector function of a vector, he proceeds to the differential calculus

of vectors. He uses essentially the derivative dyad $d\sigma = d\rho \cdot \sigma$, that is, the dyad σ. It is now easy to define divergence as the average value of the scalar $\alpha \cdot \sigma \cdot \alpha$ over the surface of a unit sphere, α being any unit vector of variable direction. This gives of course what is commonly written $\cdot\sigma$. In the same manner the average value of $\alpha \times \varphi\alpha$ (where $\varphi\alpha = \alpha \cdot \sigma$) over the surface of the unit sphere, gives the curl. This would be the same as $\times\sigma$. There is nothing new in these definitions, as they have been given in one form or another before. A development of formulas that are useful follows.

Part II of 33 pages discusses the steady motion of a solid under no forces in liquid extending to infinity. Some of the well known results are reached. Towards the latter part of the section he considers certain cases of stable motion. The general problem is rather intractable, and the author is content with stating some conclusions of the simplest case. He finds that for this simple case, the two steady motions for which the screws are parallel to the greatest and least axes of a certain ellipsoid, are stable; that steady motion for which the screw is parallel to the mean axis is unstable.

J. B. SHAW.

An Introduction to the Theory of Relativity. By L. Bolton. New York, E. P. Dutton and Company, 1921. ix + 177 pp.

This little book gives a remarkably readable and intelligible account of the theory of relativity. It is by the author of the prize-winning essay in the contest recently conducted by the SCIENTIFIC AMERICAN. The author admits at the outset the impossibility of giving any sort of adequate notion of the theory of relativity without the use of mathematical ideas and symbols. He has set himself the task, however, of presenting his material without presupposing more than elementary algebra and the elements of plane geometry. As a result he finds it necessary to introduce the reader to the notion of the differential of arc, and in so doing brings back to life the "little zeros." There is also (p. 126) a footnote implying that cones and cylinders are the only developable surfaces. Such incidental blemishes may, however, be excused in view of the book's purpose and the extraordinarily satisfactory way in which this purpose has been carried out.

J. W. YOUNG.

Geometria Analytica. Part 1. *Il Metodo delle Coordinate.* By L. Berzolari. Second edition. Milan, Ulrico Hoepli. xiii + 495 pages.

This work belongs to the excellent series of manuals of the firm of Ulrico Hoepli. It aims to define the principal systems of coordinates in space of one, two and three dimensions and to derive the principal theorems and formulas of analytic geometry connected with them. It discusses in detail Cartesian coordinates of points, of lines in a plane and of planes in space. It further deals with homogeneous coordinates, and the fundamentals of analytic projective geometry. Equations of curves and surfaces are properly treated briefly. A short but well-selected appendix on vector analysis is added to this edition. Such topics as line coordinates in space and coordinates of spheres do not fall within the scope of this

work. A few omitted topics which ought to have been included are: polar coordinates in space (radius vector and direction cosines), cylindrical coordinates and the fundamentals of intrinsic geometry.

The text is well and carefully written and, within its field, is thorough. Pedagogical requirements are borne constantly in mind and the expository form is good. American teachers and authors will find it a useful work for reference.

CHARLES SISAM.

Essai philosophique sur les Probabilités. By Pierre-Simon Laplace. I, II. Paris, Gauthier-Villars, 1921. 12 + 104 + 108 pp.

Mémoire sur la Chaleur. By MM. Lavoisier et de Laplace. Paris, Gauthier-Villars, 1920. 78 pp.

Mémoires sur l'Électromagnétisme et l'Électrodynamique. By André-Marie Ampère. Paris, Gauthier-Villars, 1921. 14 + 112 pp.

The editor of this series of reprints, entitled *Les Maîtres de la Pensée Scientifique*, pertinently remarks that the rapid scientific advances of the present time tempt us to neglect past discoveries and their authors. This neglect is almost unavoidable whenever the original papers are not within reach of the mass of scientific students. Reprints in cheap form, like the above, should meet with a hearty welcome. It is a privilege to be able to carry around in one's coat pocket the masterpieces of Laplace, Lavoisier and Ampère. Surely there is no need of enlarging upon the commanding place which each of these memoirs occupies in the history of science.

FLORIAN CAJORI.

Gruppentheorie. By Dr. L. Baumgartner. Berlin, Walter de Gruyter & Co., 1921. 120 pp.

This little book is as interesting as it is handy. It is simply written and divided into sections in a helpful way. There are four chapters: Introduction to the Group Notion, The Group Notion in Geometry, The Finite Group, The Infinite Group. The third chapter is much the longest and perhaps the most unified.

There are many illustrative examples, from theory of functions, transformations, number theory, etc., the predominance being from geometry. The literature list (p. 5) is very brief; in English only Burnside's earlier book is mentioned. The name of Lagrange is applied to the theorem, The order of a subgroup of a finite group is a factor of the order of its group (p. 7), a misnomer we are now told.* Sylow's theorem is introduced (p. 98) but a proof is appropriately omitted. If one were using this book for a first study of groups, certainly many supplementary exercises would be desirable. The little volume closes with answers to its forty-nine exercises and a very useful index.

LOUIS C. MATHEWSON.

* Carmichael, R. D., this BULLETIN, (2), vol. 27 (1922), pp. 474, 475.

NOTES

The opening number of volume 23 (1921–22) of the ANNALS OF MATHEMATICS contains the following papers: *On matrices whose elements are integers*, by Oswald Veblen and Philip Franklin; *An algorism for differential invariant theory*, by O. E. Glenn; *The general theory of cyclic-harmonic curves*, by R. E. Moritz; *More theorems on the complete quadrilateral*, by J. W. Clawson; *A theorem on cross-ratios in the geometry of inversion*, by J. L. Walsh; *The condition for an isothermal family on a surface*, by J. K. Whittemore; *The reversion of class number relations and the total representation of integers as sums of squares or triangular numbers*, by E. T. Bell; *Note on the term maximal subgroup*, by G. A. Miller; *Reducible cubic forms expressible rationally as determinants*, by L. E. Dickson; *Note on the Picard method of successive approximations*, by Dunham Jackson; *A fundamental system of covariants of the ternary cubic form*, by L. E. Dickson; *The modular theory of polyadic numbers*, by A. A. Bennett; *Some algebraic analogies in matric theory*, by A. A. Bennett; *Generalized conjugate matrices*, by Philip Franklin.

A Benjamin Pierce Instructorship in Mathematics at Harvard University (see this BULLETIN, vol. 21, page 315) is open to general competition. Applications for the year 1923–24, accompanied by the necessary papers, should reach Professor Birkhoff not later than February 15, 1923, and all communications relating to this appointment should be addressed to him.

Max Gutzmer, of Berlin, has presented the Leopoldinisch-Carolinische Akademie der Naturforscher with a sum of money as an endowment of a prize for important German methematical work. Details concerning the conditions of award will be announced later by the Academy.

Professor A. Wangerin has been elected honorary member of the Leopoldinisch-Carolinische Akademie, and has been awarded its Cothenius Medal for his mathematical work.

The Accademia dei Lincei has awarded its royal prize for astronomy for the period 1915–1920 to Professor G. Armellini, of the University of Pisa.

Dr. A. Fraenkel has been promoted to an associate professorship at the University of Marburg.

Dr. T. Kaluza has been promoted to an associate professorship at the University of Konigsberg.

Professor R. König, of the University of Tübingen, has been appointed professor of mathematics at the University of Münster.

Dr. H. Rademacher, of the University of Berlin, has been appointed to an associate professorship at the University of Hamburg.

Professor L. Hotopp, of the Hannover Technical School, has retired from active teaching.

Dr. E. Feyer has been admitted as privatdocent at the Breslau Technical School.

At the University of Geneva, Dr. D. Mirimanoff has been appointed associate professor of the calculus of probabilities, and Dr. R. Wavre associate professor of the calculus and rational mechanics.

Professor H. Beghin, of the Ecole Navale, and Dr. Soula have been appointed maîtres de conférences in mathematics at the University of Montpellier.

The following have been admitted as privatdocents at Italian universities: G. Andreoli, at the University of Naples; G. Aprile, at the University of Catania; G. Mignosi, at the University of Palermo; R. Serini and G. Usai, at the University of Pavia.

Mr. R. A. Sheets, of Denison University, has been promoted to an assistant professorship of mathematics.

Miss Evelyn Walker has been promoted to an assistant professorship of mathematics at Hunter College.

Associate Professor Gillie A. Larew, of Randolph-Macon Woman's College, has been promoted to a full professorship of mathematics.

Associate Professor H. H. Conwell, of Beloit College, has been promoted to a full professorship of mathematics.

Mr. David Hiches has been appointed professor of mathematics at Creighton University.

At the University of Wyoming, Professor C. E. Stromquist is on leave of absence on account of illness. Professor H. C. Gossard, who has for two years served as regional secretary of the Student Y. M. C. A., has returned to teaching and is acting head of the department of mathematics.

At Amherst College, Professor E. L. Dodd, on leave of absence from the University of Texas, is taking the place of Professor J. G. Hardy.

At Washington University, Mr. Theodore Doll of Northwestern University has been appointed assistant professor of mathematics.

At the University of Virginia, Mr. F. L. Brown, of Northwestern University, has been appointed assistant professor of mathematics.

At the University of Iowa, Dr. W. H. Wilson has been appointed assistant professor of mathematics, and Dr. Roscoe Woods has been appointed associate in mathematics.

The following appointments to instructorships in mathematics in American colleges and universities are announced:

University of Akron, Miss W. H. Lipscombe;
Albion College, Mr. Leon Sears;
Beloit College, Mr. R. M. Robinson;
University of California, Dr. Elsie McFarland; Southern Branch, Dr. F. C. Leonard;
University of Chicago, Dr. Mayme I. Logsdon;
Colgate University, Mr. H. A. D. Bell;
Cornell College, Mount Vernon, Mr. Edmund Ingalls;
Denison University, Mr. H. B. Lemon;
Georgia School of Technology, Mr. G. T. Trawich;
University of Georgia, Mr. A. H. Stevens;
Kenyon College, Mr. F. L. White;
Lake Forest College, Miss Marie M. Johnson;
Lehigh University, Mr. H. S. Bunn;
University of Maine, Mr. F. S. Beale, Mr. Edward Brow, Mr. W. E. Loring;
Northwestern University, Mr. R. L. Jackson, Mr. J. D. Voss;
Ottawa University, Mr. H. W. Bailey;
University of North Dakota, Mr. J. D. Leith;
University of Pittsburgh, Mr. Z. N. Holler;
Princeton University, Dr. Einar Hille, Mr. G. B. Briggs;
Sweet Briar College, Miss Mary Searle;
University of Texas, Mr. J. E. Burnam, Mr. R. Lubben;
Tufts College, Mr. P. D. Wilkins;
Wheaton College, Miss Martha F. Chadbourne;
University of Wisconsin, Miss Camilla Hayden.

Professor A. Höfler, of the University of Vienna, died February 26, 1922, at the age of sixty-eight years.

Professor E. Kötter, of the Technical School at Aachen, died January 26, 1922, at the age of sixty-two years.

Professor C. Kostka, of Insterburg, died December 28, 1921, at the age of seventy-five years.

Professor Axel Thue, of the University of Christiania, died March 7, 1922, at the age of fifty-nine years.

Professor Ernest Lebon, of the Lycée Charlemagne, Paris, died February 12, 1922, in his seventy-sixth year.

Dr. A. R. Willis, of the department of mathematical physics at the Royal College of Science, died June 23, 1922, at the age of seventy-two years.

Rear Admiral C. H. Davis, U. S. N., died at Washington December 27, 1921, at the age of seventy-six years. He was twice superintendent of the Naval Observatory.

Emeritus Professor A. A. Markov, of the University of Petrograd, died July 27, 1922, at the age of sixty-six years.

NEW PUBLICATIONS

I. HIGHER MATHEMATICS

BERGSON (H.). Durée et simultanéité. A propos de la théorie d'Einstein. Paris, Alcan, 1922. 8vo. 8 + 245 pp.

CRIVETZ (T.). La somme des angles d'un triangle rectiligne. Paris, Gauthier-Villars, 1921. 8vo. 8 pp.

CZUBER (E.). Einführung in die höhere Mathematik. 2te, durchgesehene Auflage. Leipzig, 1921.

DESARGUES (G.). Erster Entwurf eines Versuchs über die Ergebnisse des Zusammentreffens eines Kegels mit einer Ebene. Aus dem Französischen übersetzt und herausgegeben von M. Zacharias. (Ostwald's Klassiker der exakten Wissenschaften.) Leipzig, 1922.

DUPORT (—.). La loi d'attraction universelle. Paris, Gauthier-Villars, 1922. 36 pp.

DE FALCO (V.). See IAMBLICHUS.

FEHR (H.). La commission internationale de l'enseignement mathématique de 1908 a 1920. Paris, Gauthier-Villars, et Genève, George, 1921. 8vo. 40 pp.

FRICKE (R.). See KLEIN (F.).

GAUSS (C. F.). Allgemeine Flächentheorie. (Disquisitiones generales circa superficies curwas, 1827.) Deutsch herausgegeben von A. Wangerin. 5te Auflage. Leipzig, 1921.

GERLACH (J. E.). Kritik der mathematischen Vernunft. Bonn, Verlag von F. Cohen, 1922.

GIBBENS (G. E. C.). A comparison of different line-geometric representations for functions of a complex variable. (Diss., Chicago.) Menasha, Wis., Collegiate Press, 1922. 15 pp.

GRANVILLE (W. A.). The fourth dimension and the Bible. Boston, Richard G. Badger, 1922. 9 + 113 pp.

GUNTHER (R. T.). Early science in Oxford. Part 2: Mathematics. London, Oxford University Press, 1922. 101 pp.

IAMBLICHUS. Theologoumena arithmeticae. Edidit Victorius de Falco. (Bibliotheca Scriptorum Græcorum et Romanorum Teubneriana.) Leipzig, Teubner, 1922. 18 + 90 pp.

KLEIN (F.). Gesammelte mathematische Abhandlungen. Band 2: Anschauliche Geometrie, Substitutionsgruppen und Gleichungstheorie, Zur mathematischen Physik. Herausgegeben von R. Fricke und H. Vermeil. Berlin, Springer, 1922. 6 + 713 pp.

KRAMER (W.). Ueber ein besonderes geometrisches Abbildungsverfahren. (Diss., Jena.) Weida i. Th., Druck von Thomas und Hubert, 1921.

LOTZE (H.). Grundzüge der Logik und Enzyklopädie der Philosophie. Leipzig, Hirzel, 1922. 8vo. 128 pp.

MACLEOD (A.). Introduction à la géométrie non-euclidienne. Paris, Hermann, 1922. 8vo. 433 pp.

MAGGI (G. A.). Geometria del movimento. Lezioni di cinematica con

un appendice sulla geometria della massa. 2a edizione. Pisa, Spoerri, 1919. 8vo. 288 pp.

DE MENASCE (J.). See RUSSELL (B.).

MÜLLER (F. P.). Synthetischer Aufbau der Gruppe der Berührungstransformationen der Kugeln. (Diss.) Dresden, 1921.

PETROVITCH (M.). Notice sur les travaux scientifiques de M. Michel Petrovitch. Paris, Gauthier-Villars, 1922. 8vo. 9 + 152 pp.

ROGOSINSKI (W.). Neue Anwendung der Pfeifferschen Methode bei Dirichlets Teilerproblem. (Diss.) Göttingen, 1922.

RUSSELL (B.). Le mysticisme et la logique. Traduit de l'anglais par J. de Menasce. Paris, Payot, 1922. 160 pp.

SCHLESINGER (L.). Einführung in die Theorie der gewöhnlichen Differentialgleichungen auf funktionentheoretischer Grundlage. 3te, neubearbeitete Auflage. Berlin, Vereinigung wissenschaftlicher Verleger, 1922. 8 + 326 pp.

SHAW (J. B.). Vector calculus with applications to physics. New York, Van Nostrand, 1922. 6 + 314 pp. $3.50

TROPFKE (J.). Geschichte der Elementar-Mathematik in systematischer Darstellung. 2te, verbesserte und sehr vermehrte Auflage. 1ter, 2ter, und 3ter Band. Berlin, Vereinigung wissenschaftlicher Verleger, 1921–1922.

TWEEDIE (C.). James Stirling. A sketch of his life and works along with his scientific correspondence. Oxford, Clarendon Press, and New York, Oxford University Press, 1922. 12 + 213 pp.

VELTEN (A. W.). Einführung in die Theorie der elliptischen Funktionen. Teil I: Die revidierten Grundlagen der Theorie. Hannover, Helwingsche Verlagsbuchhandlung, 1922. 4 + 129 pp.

VERMEIL (H.). See KLEIN (F.).

WALFISZ (A.). Ueber die summatorischen Funktionen einiger Dirichletscher Reihen. (Diss.) Göttingen, 1922.

WANGERIN (A.). See ABEL (N. H.), GAUSS (C. F.).

ZACHARIAS (M.). See DESARGUES (G.).

II. ELEMENTARY MATHEMATICS

AHRENS (W.). Mathematische Unterhaltungen und Spiele. 3te, verbesserte Auflage. Band 1. Leipzig, 1921.

BAILEY (F. H.). See WOODS (F. S.).

BOYD (P. P.), DAVIS (J. M.), and REES (E. L.). A course in analytic geometry. New York, Van Nostrand, 1922. 10 + 252 pp.

BRAHY (E.). Exercices méthodiques de calcul intégral. 4e édition. Paris, Gauthier-Villars, 1922. 8vo. 302 pp.

DAVIS (J. M.). See BOYD (P. P.).

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FRICKE (R.). Die Hauptsätze der Differential- und Integralrechnung. Als Leitfaden zum Gebrauch bei Vorlesungen zusammengestellt. 8te Auflage. Braunschweig, 1921.

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WITTING (A.). Einführung iñ die Trigonometrie. (Mathematisch-Physikalische Bibliothek, Band 43.) Leipzig, Teubner, 1921.

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III. APPLIED MATHEMATICS

BERTHELOT (D.). La physique et la métaphysique des théories d'Einstein. Paris, Payot, 1922. 16mo. 48 pp.

BISCONCINI (G.). Elementi di matematica finanziaria e attuariale. Roma, Signorelli, 1921.

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BROWN (W.) and THOMSON (G. H.). The essentials of mental measurement. Cambridge, University Press, 1921. 10 + 216 pp.

CHARLIER (C. L.). Grundzüge der mathematischen Statistik. Hamburg, Lütke und Wulff, 1921.

CHRISTESCO (S.). La relativité et les forces dans le système cellulaire des mondes. Paris, Felix Alcan, 1921. 16mo. 290 pp.

CISOTTI (U.). Idromeccanica piana. Parte 2a. Milano, Libreria Editrice Politechnica, 1922. 8 + 219 pp.

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LAWRENCE (R. R.). Principles of alternating currents. New York, 1922. 446 pp. $4.00

LEMAIRE (J.). Notes sur les propriétés fondamentales de la matière. Liège, Société Industrielle d'Arts et Métiers, 1921. 55 pp.

LEPPER (G. H.). The generalized theory of gravitation versus the general theory of relativity. Pittsburgh, published by the author, 1921. 30 pp.

LUBSCHEZ (B. J.). Perspective. 3d edition, enlarged. New York, Van Nostrand, 1921. 125 pp. $2.00

MICHAUD (F.). Rayonnement et gravitation. Paris, Gauthier-Villars, 1922. 8vo. 5 + 61 pp.

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THEIMER (V.). Praktische Astronomie. Geographische Orts- und Zeitbestimmung. Leipzig, Teubner, 1921.

THIRY (R.). Sur les solutions multiples des problèmes d'hydrodynamique relatifs aux mouvements glissants. (Thèse, Strasbourg.) Paris, Gauthier-Villars, 1921. 4to. 112 pp.

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